

ON REPRESENTATIONS OF AFFINE HECKE ALGEBRAS OF TYPE B

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ABSTRACT. Ariki's and Grojnowski's approach to the representation theory of affine Hecke algebras of type A is applied to type B with unequal parameters to obtain – under certain restrictions on the eigenvalues of the lattice operators – analogous multiplicity-one results and a classification of irreducibles with partial branching rules as in type A .

INTRODUCTION

In this paper, the methods Ariki [1] and Grojnowski [7] developed for the representation theory of affine Hecke algebras of type A are applied to affine Hecke algebras of type B . The first section introduces the affine Hecke algebras \mathcal{H}_n – which are the main objects of interest in this paper – and their subalgebras \mathcal{H}_n^R which will be investigated in the last section. It is explained how to use Clifford theory to exploit knowledge about one algebra to obtain results about the other. The second through fourth sections closely follow Brundan and Kleshchev's paper [3] and informal lecture notes [10] by Kleshchev that are now part of his book [11]. The second section provides an affine version of the Mackey Theorem and investigates the relation between induction and coinduction functors. The third section introduces the concept of formal characters, which are the main tool in understanding finite-dimensional irreducible modules for the affine Hecke algebras of type A . The main results for \mathcal{H}_n such as irreducibility of the cosocle of certain induced modules and multiplicity-freeness of the socle of certain restricted modules are stated in the fourth section. The fifth section contains some results in the cases where the methods used in type A don't work. After providing an overview of results on the affine Hecke algebra of type A in Section 6, we then give a one-to-one correspondence between irreducibles in certain subcategories of the module category of \mathcal{H}_n^R and irreducibles in the analogous subcategories of the module category of \mathcal{H}_n^A in the last section, yielding partial branching rules in those cases.

1. THE ALGEBRAS

Fixing an algebraically closed field F of characteristic not equal to two containing deformation parameters p and q which are not roots of unity, we define the affine Hecke algebra of type B_n for $n \geq 1$ to be the associative F -algebra \mathcal{H}_n on generators

$$X_0^{\pm 1}, \dots, X_n^{\pm 1}, T_0, \dots, T_{n-1},$$

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where the T_i generate a finite Hecke algebra $\mathcal{H}_n^{\text{fin}}$ of type B_n with relations

$$\begin{aligned}
(1) \quad & (T_0 - p)(T_0 + p^{-1}) = 0 \\
(2) \quad & (T_i - q)(T_i + q^{-1}) = 0 \quad \text{for } i \geq 1 \\
(3) \quad & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for } i \geq 1 \\
(4) \quad & T_i T_j = T_j T_i \quad \text{for } |i - j| > 1 \\
(5) \quad & T_1 T_0 T_1 T_0 = T_0 T_1 T_0 T_1,
\end{aligned}$$

and the $X_i^{\pm 1}$ generate a Laurent polynomial ring \mathcal{P}_n . Those two subalgebras are subject to the mixed relations

$$\begin{aligned}
(6) \quad & T_0 X_0 T_0 = X_0 X_1 \\
(7) \quad & T_i X_j = X_j T_i \quad \text{for } j \neq i, i + 1 \\
(8) \quad & T_i X_i T_i = X_{i+1} \quad \text{for } i \geq 1.
\end{aligned}$$

For $n = 0$, we define $\mathcal{H}_0 := F[X_0^{\pm 1}]$.

This is a deformation of the group algebra of the extended affine Weyl group W_n using the weight lattice of the general orthogonal group $GO_{2n+1}(F)$, which is the subgroup of $GL_{2n+1}(F)$ respecting the orthogonal form up to a scalar. W_n is isomorphic to the semidirect product of the finite Weyl group W_n^{fin} of type B_n with generators s_0, s_1, \dots, s_{n-1} and relations

$$\begin{aligned}
s_i^2 &= 1 & \text{for all } i \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \text{for } i \geq 1 \\
s_i s_j &= s_j s_i & \text{for } |i - j| > 1 \\
s_1 s_0 s_1 s_0 &= s_0 s_1 s_0 s_1,
\end{aligned}$$

and the weight lattice of $GO_{2n+1}(F)$ which is the free abelian group on $n + 1$ generators X_0, \dots, X_n , on which W_n^{fin} acts as in the mixed relations for the affine Hecke algebra, substituting s_i for T_i . The actual affine Weyl group is the subgroup obtained from W_n by omitting the generator X_0 and adding the additional relation $s_0 X_1 s_0 = X_1^{-1}$ which in W_n can be obtained from the first of the mixed relations.

The deformation of the affine Weyl group is then naturally a subalgebra of \mathcal{H}_n generated by $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and T_0, \dots, T_{n-1} which we denote by \mathcal{H}_n^R . Here we need an additional relation

$$X_1 T_0 = T_0 X_1^{-1} + (p - p^{-1})(X_1 + 1)$$

which, in \mathcal{H}_n , can be derived from relation (6) since

$$\begin{aligned}
X_1 T_0 &= X_0^{-1} T_0 X_0 T_0 T_0 \\
&= (p - p^{-1}) X_0^{-1} T_0 X_0 T_0 + X_0^{-1} T_0 X_0 \\
&= (p - p^{-1}) X_1 + X_0^{-1} T_0^{-1} X_0 + (p - p^{-1}) \\
&= (p - p^{-1})(X_1 + 1) + T_0 X_0^{-1} X_1^{-1} X_0 \\
&= T_0 X_1^{-1} + (p - p^{-1})(X_1 + 1).
\end{aligned}$$

The commutative subalgebra generated by $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ will be denoted by \mathcal{R}_n .

For a reduced expression $w = s_{i_1} \cdots s_{i_k}$ of an element $w \in W_n^{\text{fin}}$, we define $T_w := T_{i_1} \cdots T_{i_k}$. This does not depend on the choice of reduced expression and is therefore well-defined.

In [12] Lusztig proves a general result on bases of affine Hecke algebras in the case where p and q are distinct powers of the same deformation parameter v_0 , but the proof doesn't rely on this and carries over to the general case, see [16]. In our case this result gives the following two bases for \mathcal{H}_n :

$$(1.1) \quad \left\{ X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n} T_w \mid \begin{matrix} (c_0, \dots, c_n) \in \mathbb{Z}^{n+1}, \\ w \in W_n^{\text{fin}} \end{matrix} \right\}$$

and

$$(1.2) \quad \left\{ T_w X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n} \mid \begin{matrix} (c_0, \dots, c_n) \in \mathbb{Z}^{n+1}, \\ w \in W_n^{\text{fin}} \end{matrix} \right\}.$$

All modules under consideration will be left modules that are finite-dimensional over F and the category of such modules for an F -algebra A will be denoted by $A\text{-mod}^{\text{fd}}$. For any affine Hecke algebra \mathcal{H} , Bernstein showed that its center $Z(\mathcal{H})$ is exactly the set of Laurent polynomials f in its lattice that are invariant under the action of the finite Weyl group on the lattice, see e.g. [16], §2.9. It is well-known that all irreducible representations of \mathcal{H}_n are finite-dimensional, since \mathcal{H}_n is finite-dimensional over its center, which by Dixmier's version of Schur's Lemma acts as a scalar on irreducible \mathcal{H}_n -modules. The Grothendieck group of the category $A\text{-mod}^{\text{fd}}$ will be denoted by $K(A\text{-mod}^{\text{fd}})$ and for $M \in A\text{-mod}^{\text{fd}}$ the corresponding element in $K(A\text{-mod}^{\text{fd}})$ will be written as $[M]$. For $M \in A\text{-mod}^{\text{fd}}$, we denote the socle by $\text{soc } M$ and the cosocle, i.e. the largest semisimple quotient, by $\text{cosoc } M$. If we have an automorphism ψ of A , we will, for any $M \in A\text{-mod}^{\text{fd}}$ denote by M^ψ the module obtained from M by twisting the action with ψ . This is equal to M as an abelian group but the operation of A is now via the new multiplication \diamond defined by $a \diamond m = \psi(a)m$ for $a \in A, m \in M$. The smallest integer k such that $M^{\psi^k} \cong M$ as an A -module will be called the order of ψ on M whereas the order of ψ (without specification of a module) will denote the order of ψ on A .

1.1. Clifford theory. We will use Clifford theory to move between modules for both algebras \mathcal{H}_n^R and \mathcal{H}_n . This idea to explore the interplay between different affine Hecke algebras of the same isogeny class is originally due to Xi [16] and has been worked out in detail by Ram and Ramagge [13]. In fact, Clifford theory works in a more general setting, which has been studied by Dade in [4].

Lemma 1.1. *Let n be a natural number, K an algebraically closed field of characteristic $p \geq 0$ with $p \nmid n$. Let A be a K -algebra and let B be a subalgebra of A , such that A is free as a B -module on basis $\{x^s \mid 0 \leq s \leq n-1\}$ for an invertible element x in A , and $\mathbb{Z}/n\mathbb{Z}$ -graded, i.e. $Bx^s Bx^t = Bx^{s+t}$. Let $\psi : a \mapsto x^{-1}ax$ be conjugation with x , so $\psi(B) = B$. Let $M \in A\text{-mod}^{\text{fd}}$ and let N be an irreducible B -submodule of $\text{res}_B^A M$. Then the order d of ψ on N divides n and for $k := n/d$ we have*

$$\text{res}_B^A M = \bigoplus_{j=0}^{d-1} N^{\psi^j}$$

and

$$\text{ind}_B^A N = \bigoplus_{j=0}^{k-1} M_j$$

for irreducible and pairwise non-isomorphic modules M_j . Let σ be an automorphism of A with $\sigma|_B = \text{id}_B$. All σ -conjugates of M occur as some M_j in this decomposition, so in particular, the order of σ on M_j is less than or equal to k for all $0 \leq j \leq k-1$.

Proof. Since $x^n \in B$, $N^{\psi^n} \cong N$ for any B -module N . Now let d be the smallest natural number such that $N^{\psi^d} \cong N$ and assume d does not divide n . Then for $n = qd + r$, $N^{\psi^{qd}\psi^r} \cong N$, i.e. $N^{\psi^r} \cong N$, but $r < d$, a contradiction. So, indeed d does divide n .

Let $f : N \rightarrow N^{\psi^d}$ be an isomorphism and note that then $f^j : N \rightarrow N^{\psi^{dj}}$ is also an isomorphism. In particular, since $x^n \in B$, $f^k : N \rightarrow N^{\psi^{dk}}$ is a scalar multiple of multiplication with x^{-n} , so by normalizing, we can assume that f^k is in fact multiplication with x^{-n} .

Now take any irreducible B -submodule N of M and consider $\text{ind}_B^{B'} N$, where we set $B' := \bigoplus_{0 \leq j \leq k-1} Bx^{jd}$.

Claim 1: $\text{ind}_B^{B'} N$ is a completely reducible B' -module, decomposing into a direct sum of k non-isomorphic B' -modules L_i , $i = 0, \dots, k-1$, where each L_i is isomorphic to N as B -module.

Proof of Claim 1: As a B -module $\text{ind}_B^{B'} N \cong \bigoplus_{j=0}^{k-1} x^{jd} \otimes N$. Let ζ be a primitive k -th root of unity and define L_i to be the subspace of $\text{ind}_B^{B'} N$ consisting of all elements $a_i := \sum_{j=0}^{k-1} \zeta^{ji} x^{jd} \otimes f^j(a)$ where a runs through N . It is straightforward to check that $b \cdot a_i = (ba)_i$, so L_i is a B -submodule of $\text{ind}_B^{B'} N$, giving a B -isomorphism between L_i and N . L_i is also a B' -submodule of $\text{ind}_B^{B'} N$. To see this, it suffices to show $x^d L_i \subseteq L_i$ since $B' = \bigoplus_{j=0}^{k-1} Bx^{jd}$ as a B -module. But for a in N ,

$$\begin{aligned} x^d a_i &= \sum_{j=0}^{k-1} \zeta^{ji} x^{(j+1)d} \otimes f^j(a) \\ &= \zeta^{-i} \sum_{j=0}^{k-1} \zeta^{(j+1)i} x^{(j+1)d} \otimes f^{j+1} f^{-1}(a) \\ &= \zeta^{-i} (f^{-1}(a))_i, \end{aligned}$$

which is again an element of L_i .

Now let $0 \leq i, l \leq k-1$ and suppose $i \neq l$ but $L_i \cong L_l$, i.e. there exists a B' -module isomorphism $g : L_i \rightarrow L_l$. Observe that $\text{res}_B^{B'} L_i$ is isomorphic to $\text{res}_B^{B'} L_l$ via the isomorphism $\tilde{g} : a_i \mapsto a_l$ and this is the only isomorphism up to a scalar by Schur's Lemma and the irreducibility of $\text{res}_B^{B'} L_i \cong N$. Since, if g is an isomorphism, λg is, for any $\lambda \neq 0 \in K$, also an isomorphism, we can choose g to coincide with the map \tilde{g} . But then

$$\begin{aligned} g(x^d a_i) &= \zeta^{-i} g((f^{-1}(a))_i) \\ &= \zeta^{-i} (f^{-1}(a))_l \end{aligned}$$

but also

$$\begin{aligned} g(x^d a_i) &= x^d g(a_i) \\ &= x^d a_l \\ &= \zeta^{-l} (f^{-1}(a))_l \end{aligned}$$

whence we conclude that $i = l$, contrary to our assumption. This proves Claim 1.

Now let L be one of the irreducible B' -submodules of $\text{ind}_B^{B'} N$ and consider $\text{ind}_B^A L$. The set $\{x^j \mid 0 \leq j \leq d-1\}$ forms a basis of A as a $\mathbb{Z}/d\mathbb{Z}$ -graded B' -module. The automorphism ψ leaves B' invariant as it leaves B invariant and fixes x^{jd} for $0 \leq j \leq k-1$. Therefore we can twist any B' -module with ψ and again obtain a B' -module. Since $N^{\psi^j} \not\cong N$ for $j < d$, we also have $x^j \otimes L \cong L^{\psi^j} \not\cong L$ for $j < d$.

Claim 2: $\text{ind}_B^A L$ is an irreducible A -module.

We have

$$\text{Ann}_{B'} x^i \otimes L \not\subseteq \bigcap_{\substack{0 \leq j \leq d-1 \\ j \neq i}} \text{Ann}_{B'} x^j \otimes L.$$

Otherwise an inclusion of the annihilators which are the kernels of the representations

$$\rho_{\neq i} : B' \rightarrow \text{End}_K\left(\bigoplus_{\substack{0 \leq j \leq d-1 \\ j \neq i}} x^j \otimes L\right)$$

and

$$\rho_i : B' \rightarrow \text{End}_K(x^i \otimes L)$$

would give rise to a projection on the side of the images, which cannot happen since the $x^j \otimes L$ are pairwise non-isomorphic irreducible B' -modules.

Let $0 \neq a = \sum_{j=0}^{d-1} x^j \otimes a_j$ with $a_j \in L$ be an element of $\text{ind}_{B'}^A L$. Suppose $a_i \neq 0$. Then the left ideal $\text{Ann}_{B'} x^i \otimes a_i$ contains the maximal two-sided ideal $\text{Ann}_{B'} x^i \otimes L$. If it also contained $\bigcap_{\substack{0 \leq j \leq d-1 \\ j \neq i}} \text{Ann}_{B'} x^j \otimes L$, it would contain the two-

sided ideal $\text{Ann}_{B'} x^i \otimes L + \bigcap_{\substack{0 \leq j \leq d-1 \\ j \neq i}} \text{Ann}_{B'} x^j \otimes L$, which, by the above, is strictly

larger than $\text{Ann}_{B'} x^i \otimes L$ and thus equals B' by the maximality of $\text{Ann}_{B'} x^i \otimes L$.

Therefore we can find a non-zero element such that

$$ya = yx^i \otimes a_i = x^i \otimes y'a_i \neq 0,$$

where $y' = x^{-i}yx^i$ and therefore we have an element $0 \neq b_i := y'a_i \in L$ such that $x^i \otimes b_i \in B'a$. Thus, $x^i \otimes L$ is a B' -submodule of $B'a$ and $x^j \otimes L$ is contained in Aa for all $j = 0, \dots, d-1$. This proves Claim 2.

Now, what is left for us to prove is that the $M_i := \text{ind}_{B'}^A L_i$ are also pairwise non-isomorphic and that all conjugates of M by distinct powers of σ occur in the decomposition.

Frobenius reciprocity gives

$$\begin{aligned} \text{Hom}_A(M_i, M_l) &\cong \text{Hom}_{B'}(L_i, \bigoplus_{j=0}^{d-1} x^j \otimes L_l) \\ &\cong \text{Hom}_{B'}(L_i, L_l) \oplus \bigoplus_{j=1}^{d-1} \text{Hom}_{B'}(L_i, x^j \otimes L_l). \end{aligned}$$

$\text{Hom}_{B'}(L_i, L_l) = 0$ for $i \neq l$, according to Claim 1, while

$$\bigoplus_{j=1}^{d-1} \text{Hom}_{B'}(L_i, x^j \otimes L_l) \hookrightarrow \bigoplus_{j=1}^{d-1} \text{Hom}_B(\text{res}_B^{B'} L_i, \text{res}_B^{B'} x^j \otimes L_l)$$

but the latter is just $\bigoplus_{j=1}^{d-1} \text{Hom}_B(N, N^{\psi^j})$ which is zero by hypothesis. So the M_i are pairwise non-isomorphic.

Now we have $\text{ind}_B^A N \cong \bigoplus_{i=0}^{k-1} M_i$ and $\text{res}_B^A M_i \cong \bigoplus_{j=0}^{d-1} x^j \otimes N$. In particular, this implies the asserted decomposition of M , since M is isomorphic to one of the M_i by Frobenius reciprocity. As σ fixes B pointwise, $S^{\sigma^l} = S$ for all B -modules S and all $l \in \mathbb{Z}$. So

$$\begin{aligned} K &\cong \text{Hom}_B(N, \text{res}_B^A M_i) \\ &\cong \text{Hom}_B(N, \text{res}_B^A M_i^{\sigma^l}) \\ &\cong \text{Hom}_A\left(\bigoplus_{j=0}^{k-1} M_j, M_i^{\sigma^l}\right) \end{aligned}$$

gives an inclusion of $\{M_i^{\sigma^l} \mid l \in \mathbb{Z}\}$ into $\{M_j \mid 0 \leq j \leq k-1\}$ for a fixed i . Therefore the order of σ on M_i is at most k for any i . \square

Now, in order to apply this to our situation, observe that $X_0^2 \prod_{i=1}^n X_i$ is central in \mathcal{H}_n . To verify this, note that every generator T_i commutes with all but X_i and X_{i+1} . But, for $i \geq 1$, $T_i X_i X_{i+1} = X_{i+1} T_i^{-1} X_{i+1} = X_i X_{i+1} T_i$. Similarly, $X_0^2 X_1 T_0 = X_0 X_1 T_0^{-1} X_0 X_1 = T_0 X_0^2 X_1$, so $X_0^2 \prod_{i=1}^n X_i$ is indeed central and invertible and therefore acts as a nonzero scalar μ_M on an irreducible module $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$. Thus, the action of \mathcal{H}_n on an irreducible module $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ factors over the quotient algebra $\mathcal{H}_n^{\mu_M} := \mathcal{H}_n / (X_0^2 \prod_{i=1}^n X_i - \mu_M)$.

Lemma 1.2. $\mathcal{H}_n^{\mu_M}$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathcal{H}_n^R -module with basis $\{1, X_0\}$.

Proof. We first show that \mathcal{H}_n^R can be identified with a subalgebra of $\mathcal{H}_n^{\mu_M}$. Since $X_0^2 \prod_{i=1}^n X_i$ is central the ideal generated by $X_0^2 \prod_{i=1}^n X_i - \mu_M$ is already contained in and thus equal to the right ideal $(X_0^2 \prod_{i=1}^n X_i - \mu_M) \mathcal{H}_n$ which is generated over F by

$$\begin{aligned} & (X_0^2 \prod_{i=1}^n X_i - \mu_M) X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n} T_w \\ &= (X_0^{c_0+2} X_1^{c_1+1} \cdots X_n^{c_n+1} - \mu_M X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n}) T_w \end{aligned}$$

for $(c_0, \dots, c_n) \in \mathbb{Z}^{n+1}$, and $w \in W_n^{\text{fin}}$.

No finite linear combination of those can have degree 0 in X_0 , therefore the intersection of the ideal generated by $(X_0^2 \prod_{i=1}^n X_i - \mu_M)$ with \mathcal{H}_n^R is zero and we can view \mathcal{H}_n^R as a subalgebra of $\mathcal{H}_n^{\mu_M}$.

As X_0 commutes with all generators X_j and all generators T_j except T_0 and

$$\begin{aligned} T_0 X_0 &= T_0^{-1} X_0 + (p - p^{-1}) X_0 \\ &= X_0 T_0 X_1^{-1} + (p - p^{-1}) X_0 \\ &= X_0 (T_0 X_1^{-1} + (p - p^{-1})) \end{aligned}$$

we see that $\mathcal{H}_n^R X_0 \subseteq X_0 \mathcal{H}_n^R$, which by the same argument as for \mathcal{H}_n^R above has no nontrivial intersection with the ideal generated by $(X_0^2 \prod_{i=1}^n X_i - \mu_M)$ and can therefore be viewed as contained in $\mathcal{H}_n^{\mu_M}$. Certainly $\mathcal{H}_n^R \cap \mathcal{H}_n^R X_0 = \{0\}$, so we have an \mathcal{H}_n^R -submodule of $\mathcal{H}_n^{\mu_M}$ which is isomorphic to $\mathcal{H}_n^R \oplus \mathcal{H}_n^R X_0$. But $X_0 \mathcal{H}_n^R X_0 \subseteq \mathcal{H}_n^R + (X_0^2 \prod_{i=1}^n X_i - \mu_M) \mathcal{H}_n$, as we see by considering that $X_0^2 = \mu_M (\prod_{i=1}^n X_i^{-1}) \in \mathcal{H}_n^{\mu_M}$ and $X_0 T_0 X_0 = T_0^{-1} X_0^2 X_1$. Thus $\mathcal{H}_n^{\mu_M} \cong \mathcal{H}_n^R \oplus \mathcal{H}_n^R X_0$ as a left $\mathbb{Z}/2\mathbb{Z}$ -graded \mathcal{H}_n^R -module. \square

Set

$$\psi : \mathcal{H}_n \rightarrow \mathcal{H}_n : h \mapsto X_0^{-1} h X_0$$

and

$$\begin{aligned} \sigma : \mathcal{H}_n \rightarrow \mathcal{H}_n : \quad & T_i \mapsto T_i \\ & X_0 \mapsto -X_0 \\ & X_i \mapsto X_i \quad \text{for } i \geq 1. \end{aligned}$$

Both automorphisms leave the ideal generated by $(X_0^2 \prod_{i=1}^n X_i - \mu_M)$ invariant. For ψ , this follows from the commutativity of \mathcal{P}_n and for σ from the fact that $(-X_0)^2 = X_0^2$. Thus ψ and σ define automorphisms on $\mathcal{H}_n^{\mu_M}$. Then ψ leaves \mathcal{H}_n^R invariant since

$$X_0^{-1} T_0 X_0 = X_1 T_0^{-1}$$

and σ fixes \mathcal{H}_n^R pointwise since it is the identity on the generators of \mathcal{H}_n^R . Applying Lemma 1.1, the restriction to \mathcal{H}_n^R of an irreducible $\mathcal{H}_n^{\mu_M}$ -module M splits only if $M^\sigma \cong M$. Recall that $\text{char } F \neq 2$.

Lemma 1.3. $M^\sigma \cong M$ only if -1 occurs as an eigenvalue for some X_j , $j = 1, \dots, n$.

Proof. The element $X_0 \prod_{1 \leq i \leq n} (1 + X_i)$ is central in \mathcal{H}_n as

$$\begin{aligned} X_0(1 + X_1)T_0 &= X_0T_0 + X_0X_1T_0 \\ &= T_0^{-1}X_0X_1 + T_0X_0 + (p - p^{-1})X_0X_1 \\ &= T_0(X_0 + X_0X_1) \end{aligned}$$

and

$$\begin{aligned} (X_i + 1)(X_{i+1} + 1)T_i &= X_iX_{i+1}T_i + X_iT_i + X_{i+1}T_i + T_i \\ &= T_iX_iX_{i+1} + T_i^{-1}X_{i+1} + T_iX_i \\ &\quad + (q - q^{-1})X_{i+1} + T_i \\ &= T_i(X_i + 1)(X_{i+1} + 1) \end{aligned}$$

and all other factors commute with T_i anyway. If -1 does not occur as an eigenvalue for any of the X_i , $1 \leq i \leq n$, $X_0 \prod_{1 \leq i \leq n} (1 + X_i)$ acts by a nonzero scalar on M and by its negative on M^σ , so the two are not isomorphic. \square

If -1 occurs as an eigenvalue of some X_i on M , it can indeed happen that $M^\sigma \cong M$ but this will not always be the case.

Example 1.4. Consider the 2-dimensional module M for \mathcal{H}_1 on which the generators T_0, X_0, X_1 act by the matrices

$$\begin{pmatrix} p & 0 \\ 0 & -p^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & a_0 \\ a_0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively. This is obviously irreducible but splits upon restriction to \mathcal{H}_1^R , the isomorphism between M and M^σ being given by multiplication with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. On the other hand, there is a two-dimensional representation for \mathcal{H}_2 where T_0 and T_1 act as

$$\begin{pmatrix} \frac{(p-p^{-1})q^2}{(q^2+1)} & \frac{p^4q^2+q^4p^2+p^2+q^2}{pq^2(p^2-1)(1+q^2)} \\ \frac{q^2(p-p^{-1})}{(q^2+1)} & \frac{(p-p^{-1})}{(q^2+1)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix}$$

and X_0, X_1 and X_2 act as

$$\begin{pmatrix} a_0 & 0 \\ 0 & -a_0q^2 \end{pmatrix}, \quad \begin{pmatrix} -q^2 & 0 \\ 0 & -q^{-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively, which is also irreducible and which remains irreducible when restricted to \mathcal{H}_2^R .

Since our main goal is to understand finite dimensional irreducible modules for affine Hecke algebras by looking at the action of the lattice, we will generally work with the algebra \mathcal{H}_n as opposed to the algebra \mathcal{H}_n^R . We do not know whether the action of the lattice uniquely determines irreducibles for \mathcal{H}_n but at least it does in all examples we have been able to compute, contrary to \mathcal{H}_n^R where the above is an immediate counterexample.

1.2. Notation and computations. The affine Hecke algebra of type \tilde{A}_n using the weight lattice of GL_n is naturally embedded in \mathcal{H}_n as the subalgebra generated by T_1, \dots, T_{n-1} and $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ which will be denoted by \mathcal{H}_n^A . A lot of work has been done on this algebra, upon which we will heavily rely in the following. We will review the theory in type A when needed and point out similarities and differences along the way.

To simplify notation, we now introduce some abbreviations. A simultaneous eigenvector for a set of lattice operators X_{i_1}, \dots, X_{i_r} with respective eigenvalues a_{i_1}, \dots, a_{i_r} will be called an $(a_{i_1}, \dots, a_{i_r})$ -eigenvector for X_{i_1}, \dots, X_{i_r} . Analogously, we will refer to the subspace of all $(a_{i_1}, \dots, a_{i_r})$ -eigenvectors for X_{i_1}, \dots, X_{i_r} as the $(a_{i_1}, \dots, a_{i_r})$ -eigenspace for X_{i_1}, \dots, X_{i_r} . If the specification “for X_{i_1}, \dots, X_{i_r} ” is omitted, the tuple $(a_{i_1}, \dots, a_{i_r})$ necessarily consists of as many entries as there are lattice operators and the lattice operators are taken in order, i.e. the (a_0, a_1, \dots, a_n) -eigenspace in a module $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ denotes the (a_0, a_1, \dots, a_n) -eigenspace for X_0, X_1, \dots, X_n . The same conventions will be used for generalized eigenvectors and eigenspaces.

We will write $T_k T_{k+1} \cdots T_l$ for $k \leq l$ or $T_k T_{k-1} \cdots T_l$ for $k \geq l$ and $T_k \cdots T_0 \cdots T_l$ as $T_{k,l}$ and $T_{k,0,l}$ respectively, and adopt the analogous convention for $s_{k,l}$ and $s_{k,0,l}$.

Next, we state some technical lemmas which are easily checked by direct computations using the defining relations.

Lemma 1.5. *Let $N \in \mathcal{H}_n\text{-mod}^{\text{fd}}$.*

- (i) *Let $j \geq 1$ and let $u \in N$ be an (a, b) -eigenvector for X_j, X_{j+1} where $a, b \in F$ satisfy $a^{-1}b \notin \{q^{\pm 2}\}$. Then $(T_j - a^{-1}bT_j^{-1})u$ is a (b, a) -eigenvector for X_j, X_{j+1} .*
- (ii) *Let $u \in N$ be an (a_0, a) -eigenvector for X_0, X_1 , where $a \notin \{p^2, 1\}$. Then $(T_0 - aT_0^{-1})u$ is an (a_0a, a^{-1}) -eigenvector for X_0, X_1 .*

The restrictions on a and b guarantee that the vectors $(T_j - a^{-1}bT_j^{-1})u$ resp. $(T_0 - aT_0^{-1})u$ are nonzero.

We will now consider the behavior of elements in \mathcal{H}_n when we move lattice elements from one side to the other but we first need to define the Bruhat order on W_n^{fin} : For $x, y \in W_n^{\text{fin}}$, $x < y$ if and only if there exists a reduced expression $y = u_1 \cdots u_k$ where $u_j \in \{s_i \mid i \in I\}$ for $1 \leq j \leq k$ and a subsequence $1 \leq m_1 < \cdots < m_l \leq k$ such that $x = u_{m_1} \cdots u_{m_l}$. This defines a partial order on W_n^{fin} which is compatible with the length function. The following lemma is also straightforward.

Lemma 1.6. *Let $w \in W_n^{\text{fin}}$ and $0 \leq i \leq n$. Then*

$$T_w X_i \in X_{w(i)} T_w + \sum_{\tilde{w} < w} \mathcal{P}_n T_{\tilde{w}}$$

where $X_{w^{-1}(i)}$ is the element $w^{-1}X_iw$ when we look at the action of the finite Weyl group on the lattice.

2. MACKEY FILTRATION AND DUALITY

In a finite Coxeter group, a parabolic subgroup is generally defined as being conjugate to a standard parabolic subgroup, which is the subgroup generated by a subset of the Coxeter generators s_i . For W_n^{fin} this means we take a subset

$$\begin{aligned} I = \{i_1, i_1 + 1, \dots, i_1 + r_1 - 1, i_2, \dots, i_2 + r_2 - 1, \dots, i_l, \dots, i_l + r_l - 1 \\ \mid i_k + r_k < i_{k+1} \quad \forall 1 \leq k \leq l - 1\} \\ \subseteq \{0, 1, \dots, n - 1\} \end{aligned}$$

and obtain $W_I^{\text{fin}} := \langle s_i \mid i \in I \rangle$ which is isomorphic to $W_{r_1}^{\text{fin}} \times S_{r_2} \times \cdots \times S_{r_l}$ if $i_1 = 0$ and to $S_{r_1} \times S_{r_2} \times \cdots \times S_{r_l}$ if $0 \notin I$.

We generalize this concept to (extended) affine Weyl groups not by taking a subset of the Coxeter generators which would yield finite Coxeter groups of infinite index in the affine Weyl group — making induction difficult to handle — but by taking the semidirect product of a (standard) parabolic subgroup of the finite Weyl

group with the full translation lattice. Thus we define $W_I := \langle s_i, X_j \mid i \in I, j \in \{0, \dots, n\} \rangle$.

Analogously, the parabolic subalgebra $\mathcal{H}_I^{\text{fin}}$ of $\mathcal{H}_n^{\text{fin}}$ is defined as the subalgebra generated by $\{T_i \mid i \in I\}$ and the parabolic subalgebra \mathcal{H}_I of \mathcal{H}_n is the subalgebra generated by $\{T_i, X_j^{\pm 1} \mid i \in I, j \in \{0, \dots, n\}\}$. We will also write (m_0, \dots, m_l) with $m_0 \geq 0$ and $m_i \geq 1$ for $i \geq 1$ where $m_0 + \dots + m_l = n$ for

$$I = \{0, 1, \dots, m_0 - 1, m_0 + 1, \dots, m_0 + m_1 - 1, \dots, \sum_{i=0}^{l-1} m_i + 1, \dots, n - 1\}$$

and write $\mathcal{H}_{m_0, \dots, m_l}$ for \mathcal{H}_I which is isomorphic to $\mathcal{H}_{m_0} \otimes \mathcal{H}_{m_1}^A \otimes \dots \otimes \mathcal{H}_{m_l}^A$. In case $m_0 = 0$, $\mathcal{H}_{0, m_1, \dots, m_l} \cong F[X_0^{\pm 1}] \otimes \mathcal{H}_{m_1, \dots, m_l}^A \cong F[X_0^{\pm 1}] \otimes \mathcal{H}_{m_1}^A \otimes \mathcal{H}_{m_2}^A \otimes \dots \otimes \mathcal{H}_{m_l}^A$ denotes the tensor product of $F[X_0^{\pm 1}]$ with the parabolic subalgebra of type A corresponding to

$$I = \{1, \dots, m_1 - 1, m_1 + 1, \dots, m_1 + m_2 - 1, \dots, \sum_{i=1}^{l-1} m_i + 1, \dots, n - 1\}.$$

We will generally abbreviate induction and restriction functors between parabolic subalgebras as $\text{ind}_J^I := \text{ind}_{\mathcal{H}_J}^{\mathcal{H}_I}$, $\text{res}_J^I := \text{res}_{\mathcal{H}_J}^{\mathcal{H}_I}$, $\text{ind}_{m_0, \dots, m_l}^{n_0, \dots, n_k} := \text{ind}_{\mathcal{H}_{m_0, \dots, m_l}}^{\mathcal{H}_{n_0, \dots, n_k}}$ and $\text{res}_{m_0, \dots, m_l}^{n_0, \dots, n_k} := \text{res}_{\mathcal{H}_{m_0, \dots, m_l}}^{\mathcal{H}_{n_0, \dots, n_k}}$ for $J \subseteq I$ or $(m_0, \dots, m_l) \subseteq (n_0, \dots, n_k)$ respectively. If we induce directly from a parabolic subalgebra of type A , we will always use the full expression.

For a parabolic subgroup W_I^{fin} of W_n^{fin} , there are distinguished left and right coset representatives of minimal length, the sets of which will be denoted by D_I and D_I^{-1} respectively. For parabolic subgroups W_I^{fin} and W_J^{fin} of W_n^{fin} , $D_{I,J} := D_I^{-1} \cap D_J$ is then the set of distinguished minimal length $(W_I^{\text{fin}}, W_J^{\text{fin}})$ -double coset representatives. An account of this, including the following three properties of distinguished double coset representatives, can be found in [6], Chapter 2.1.

- (i) For $x \in D_{I,J}$, $W_I^{\text{fin}} \cap x W_J^{\text{fin}} x^{-1} =: W_{I \cap xJ}^{\text{fin}}$ and $x^{-1} W_I^{\text{fin}} x \cap W_J^{\text{fin}} =: W_{x^{-1}I \cap J}^{\text{fin}}$ are parabolic subgroups of W_n^{fin} . This defines subsets $I \cap xJ$ and $x^{-1}I \cap J$ of $\{0, 1, \dots, n - 1\}$.
- (ii) For $x \in D_{I,J}$, the map

$$\begin{aligned} W_{I \cap xJ}^{\text{fin}} &\rightarrow W_{x^{-1}I \cap J}^{\text{fin}} \\ w &\mapsto x^{-1}wx \end{aligned}$$

defines a length preserving isomorphism.

- (iii) For $x \in D_{I,J}$, every $w \in W_I^{\text{fin}} x W_J^{\text{fin}}$ can be written as $w = uxv$ for unique elements $u \in W_I^{\text{fin}}$ and $v \in W_J^{\text{fin}} \cap D_{x^{-1}I \cap J}^{-1}$. Moreover, $W_J^{\text{fin}} \cap D_{x^{-1}I \cap J}^{-1}$ is the set of minimal length right coset representatives of $W_{x^{-1}I \cap J}^{\text{fin}}$ in W_J^{fin} .

Lemma 2.1. *For $x \in D_{I,J}$, the subspace $\mathcal{H}_I^{\text{fin}} T_x \mathcal{H}_J^{\text{fin}}$ has basis $\{T_w \mid w \in W_I^{\text{fin}} x W_J^{\text{fin}}\}$.*

Proof. The proof of this is based on a standard argument using coset representatives and will be omitted. \square

Lemma 2.2. *For $x \in D_{I,J}$ the subspace $\mathcal{H}_I T_x \mathcal{H}_J^{\text{fin}}$ of \mathcal{H}_n has basis*

$$B_{I,J}^x := \{X_0^{c_0} \cdots X_n^{c_n} T_w \mid (c_0, \dots, c_n) \in \mathbb{Z}^{n+1}, w \in W_I^{\text{fin}} x W_J^{\text{fin}}\}.$$

Moreover, as a vector space,

$$\mathcal{H}_n = \bigoplus_{x \in D_{I,J}} \mathcal{H}_I T_x \mathcal{H}_J^{\text{fin}}.$$

Proof. Analogous to [3], Lemma 2.5. \square

We now fix some total order \prec refining the Bruhat order $<$ on $D_{I,J}$. For $x \in D_{I,J}$, set

$$(2.1) \quad \begin{aligned} \mathcal{B}_{\preceq x} &= \bigoplus_{y \in D_{I,J}, y \preceq x} \mathcal{H}_I T_y \mathcal{H}_J^{\text{fin}}, \\ \mathcal{B}_{\prec x} &= \bigoplus_{y \in D_{I,J}, y \prec x} \mathcal{H}_I T_y \mathcal{H}_J^{\text{fin}}, \\ \mathcal{B}_x &= \mathcal{B}_{\preceq x} / \mathcal{B}_{\prec x}. \end{aligned}$$

This defines a filtration of \mathcal{H}_n as an $(\mathcal{H}_I, \mathcal{H}_J)$ -bimodule, since it follows from Lemma 1.6 that we can move lattice elements from the right to the left and only create terms that are smaller in the Bruhat order and therefore lower in the filtration. Property (ii) of double coset representatives above implies that for each $x \in D_{I,J}$, there exists an algebra isomorphism

$$\begin{aligned} \phi_{x^{-1}} : \mathcal{H}_{I \cap xJ} &\rightarrow \mathcal{H}_{x^{-1}I \cap J} : T_w \mapsto T_{x^{-1}wx} \\ X_j &\mapsto X_{x^{-1}(j)} \end{aligned}$$

for $0 \leq j \leq n-1$ and $w \in W_{I \cap xJ}^{\text{fin}}$. Note that $X_{x^{-1}(j)}$ is not necessarily of the form X_i but can be the product of several polynomial generators. For $N \in \mathcal{H}_{x^{-1}I \cap J}\text{-mod}^{\text{fd}}$, ${}^x N$ will denote the $\mathcal{H}_{I \cap xJ}$ -module obtained by pulling back the action through $\phi_{x^{-1}}$.

Now we can prove an affine version of the Mackey theorem, which differs from “classical” Mackey theorems in that it does not give a direct decomposition but only a filtration.

Theorem 2.3. (“Mackey Theorem”) *Let $M \in \mathcal{H}_J\text{-mod}^{\text{fd}}$. Then $\text{res}_I^n \text{ind}_J^n M$ admits a filtration with subquotients isomorphic to $\text{ind}_{I \cap xJ}^I ({}^x \text{res}_{x^{-1}I \cap J}^J M)$, one for each $x \in D_{I,J}$. The subquotients can be taken in any order refining the Bruhat order on $D_{I,J}$, in particular, since the double coset representative 1 is the smallest element in the ordering, $\text{ind}_{I \cap J}^I \text{res}_{I \cap J}^J M$ appears as a submodule.*

Proof. We already have a filtration of \mathcal{H}_n as an $(\mathcal{H}_I, \mathcal{H}_J)$ -bimodule given in (2.1). Thus $\text{res}_I^n \text{ind}_J^n M = \mathcal{H}_I \mathcal{H}_n \otimes_{\mathcal{H}_J} M$ inherits a filtration with subquotients isomorphic to $\mathcal{B}_x \otimes_{\mathcal{H}_J} M$, the $x \in D_{I,J}$ taken in any order refining the Bruhat order on $D_{I,J}$. Now

$$\begin{aligned} \text{ind}_{I \cap xJ}^I ({}^x \text{res}_{x^{-1}I \cap J}^J M) &= \mathcal{H}_I \otimes_{\mathcal{H}_{I \cap xJ}} {}^x (\mathcal{H}_{x^{-1}I \cap J} \otimes_{\mathcal{H}_J} M) \\ &\cong \mathcal{H}_I \otimes_{\mathcal{H}_{I \cap xJ}} {}^x \mathcal{H}_J \otimes_{\mathcal{H}_J} M, \end{aligned}$$

thus it suffices to show that $\mathcal{H}_I \otimes_{\mathcal{H}_{I \cap xJ}} {}^x \mathcal{H}_J \cong \mathcal{B}_x$. In order to do this, we define a bilinear map

$$\begin{aligned} \mathcal{H}_I \times {}^x \mathcal{H}_J &\rightarrow \mathcal{B}_x \\ (h, h') &\mapsto hT_x h' + \mathcal{B}_{\prec x}. \end{aligned}$$

Since, for $w \in \mathcal{H}_{I \cap xJ}$, $T_w T_x = T_{wx} = T_{xx^{-1}wx} = T_x T_{x^{-1}wx}$, and for all j , we have $X_j T_x = T_{x^{-1}}^{-1} T_{x^{-1}} X_j T_x = T_{x^{-1}}^{-1} X_{x^{-1}(j)} \in T_x X_{x^{-1}(j)} + \mathcal{B}_{\prec x}$, this map is $\mathcal{H}_{I \cap xJ}$ -balanced and therefore induces a map $\mathcal{H}_I \otimes_{\mathcal{H}_{I \cap xJ}} {}^x \mathcal{H}_J \rightarrow \mathcal{B}_x$.

By Lemma 2.1 and property (iii) above it, a basis of $\mathcal{H}_I \otimes_{\mathcal{H}_{I \cap xJ}} {}^x \mathcal{H}_J$ is given by

$$\{X_0^{c_0} \cdots X_n^{c_n} T_u \otimes T_v \mid (c_0, \dots, c_n) \in \mathbb{Z}^{n+1}, u \in W_I^{\text{fin}}, v \in W_J^{\text{fin}} \cap D_{x^{-1}I \cap J}^{-1}\},$$

the elements of which map to a basis of \mathcal{B}_x by Lemma 2.2, whence the map is actually an isomorphism. \square

In general, for $N \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ and $M \in \mathcal{H}_I\text{-mod}^{\text{fd}}$, by Frobenius reciprocity

$$(2.2) \quad \text{Hom}_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_I} M, N) \cong \text{Hom}_{\mathcal{H}_I}(M, \text{res}_{\mathcal{H}_I}^{\mathcal{H}_n} N)$$

and

$$(2.3) \quad \text{Hom}_{\mathcal{H}_I}(\text{res}_{\mathcal{H}_I}^{\mathcal{H}_n} N, M) \cong \text{Hom}_{\mathcal{H}_n}(N, \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, M)).$$

We would like to express the coinduced module $\text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, M)$ in terms of an induced module. Hence, for the rest of this section, we fix a subset I of $\{0, 1, \dots, n-1\}$. Let d be the longest element of $D_{I,I}$.

Lemma 2.4. *Let $I = (m_1, \dots, m_l)$. Then the longest double coset representative d in $D_{I,I}$ is an involution and $I \cap dI = I \cap d^{-1}I = I$.*

Proof. Let w_0 be the longest element of W_n^{fin} . This element has to map the short root X_1 in the basis of the root system to a short root in the inverse of this basis which is $X_1^{-1}, X_2^{-1}X_1, \dots, X_n^{-1}X_{n-1}$. The only short root there is X_1^{-1} and since w_0 acts as an isometry it follows that it sends each root to its negative, hence also each X_i to X_i^{-1} . Set $k_i = \sum_{j=1}^i m_j$ and let $w_{0,I}$ be the longest element in

$$W_I^{\text{fin}} \cong W_{m_1} \times S_{m_2} \times \dots \times S_{m_l},$$

which is the element sending X_1, \dots, X_{m_1} to their inverses as above and reversing the orders of $X_{k_i+1}, \dots, X_{k_i+m_{i+1}}$ for $1 \leq i \leq l-1$, as the element reversing the order of the numbers $1, \dots, m_i$ is the longest element in the symmetric group S_{m_i} .

For the longest distinguished left coset representative \tilde{d} in D_I , we have $w_0 = \tilde{d}w_{0,I}$ by the additivity of lengths for distinguished coset representatives. Since $w_{0,I}$ is equal to its inverse, we obtain $\tilde{d} = w_0w_{0,I}$. Computing the action of \tilde{d} on the X_i , we see that it leaves X_1, \dots, X_{m_1} invariant and maps the ordered sets $(X_{k_i+1}, \dots, X_{k_i+m_{i+1}})$ to $(X_{k_i+m_{i+1}}^{-1}, \dots, X_{k_i+1}^{-1})$ for $1 \leq i \leq l-1$. From this presentation it is easy to see that \tilde{d}^{-1} is equal to \tilde{d} , whence it is also the longest distinguished right coset representative and therefore the longest element d in $D_{I,I}$. By direct computation it follows that $ds_id = s_i$ for $0 \leq i \leq m_1-1$ and $ds_{k_i+j}d = s_{k_{i+1}+1-j}$ for $1 \leq j \leq m_{i+1}$ and $1 \leq i \leq l-1$, which shows that $I \cap dI = I \cap d^{-1}I = I$. \square

By property (ii) of distinguished double coset representatives, there is an isomorphism

$$(2.4) \quad \phi = \phi_{d^{-1}} : \mathcal{H}_I \rightarrow \mathcal{H}_I,$$

and for $M \in \mathcal{H}_I\text{-mod}^{\text{fd}}$, we denote by dM the \mathcal{H}_I -module obtained by twisting the action with ϕ .

We will need a homomorphism $\theta : \mathcal{H}_n \rightarrow {}^d\mathcal{H}_I$ of $(\mathcal{H}_I, \mathcal{H}_I)$ -bimodules where the right action of \mathcal{H}_I on ${}^d\mathcal{H}_I$ is the usual (untwisted) one. This homomorphism is given by first projecting $\mathcal{H}_n \rightarrow \mathcal{B}_d$ in (2.1) and then applying the isomorphism of $\mathcal{B}_d \rightarrow \mathcal{H}_I \otimes_{\mathcal{H}_I} {}^d\mathcal{H}_I \cong {}^d\mathcal{H}_I$ given in the proof of Theorem 2.3. Explicitly, this homomorphism is given by

$$\theta(XT_w) = \begin{cases} \phi(X)T_{d^{-1}w} & \text{if } w \in dW_I^{\text{fin}}, \\ 0 & \text{otherwise,} \end{cases}$$

for $X \in \mathcal{P}_n, w \in W_n^{\text{fin}}$. Then the following holds.

Lemma 2.5. *The map*

$$\begin{aligned} f : \mathcal{H}_n &\rightarrow \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, {}^d\mathcal{H}_I) \\ h &\mapsto (h\theta : t \mapsto \theta(th)) \end{aligned}$$

is an isomorphism of $(\mathcal{H}_n, \mathcal{H}_I)$ -bimodules.

Proof. First, we need to show that f is an $(\mathcal{H}_n, \mathcal{H}_I)$ -bimodule homomorphism. So, we check $f(h)(t) = \theta(th) = h\theta(t) = hf(1)(t)$ for $h \in \mathcal{H}_n$ and $f(h')(t) = \theta(th') = \theta(t)h' = f(1)h'(t)$ for $h' \in \mathcal{H}_I$.

Since, as left \mathcal{H}_I -module, ${}^d\mathcal{H}_I$ is isomorphic to \mathcal{H}_I and \mathcal{H}_n is a free left \mathcal{H}_I -module on basis $\{T_w \mid w \in D_I^{-1}\}$, the set $K := \{\psi_w \mid w \in D_I^{-1}\}$, where $\psi_w : \mathcal{H}_n \xrightarrow{d} \mathcal{H}_I : \psi_w(T_u) = \delta_{u,w}$ for $u \in D_I^{-1}$, is a basis for $\text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, {}^d\mathcal{H}_I)$ as a free right \mathcal{H}_I -module.

So, we'll be done if we can give a basis for \mathcal{H}_n as free right \mathcal{H}_I -module that is mapped to K . But since $\mathcal{H}_I = \mathcal{H}_I^{\text{fin}}\mathcal{P}_n$, a basis for $\mathcal{H}_I^{\text{fin}}$ as free right $\mathcal{H}_I^{\text{fin}}$ -module is automatically a basis for \mathcal{H}_n as free right \mathcal{H}_I -module. Therefore, we study the restrictions $\theta' := \theta|_{\mathcal{H}_n^{\text{fin}}}$ and $f' := f|_{\mathcal{H}_n^{\text{fin}}}$ to $\mathcal{H}_n^{\text{fin}}$ of the above homomorphisms and want to construct a basis for $\mathcal{H}_n^{\text{fin}}$ as free right $\mathcal{H}_I^{\text{fin}}$ -module such that the basis elements map to the $\psi'_w := \psi_w|_{\mathcal{H}_n^{\text{fin}}}$. A basis like this can be found if we can show that f' is an isomorphism of $(\mathcal{H}_n^{\text{fin}}, \mathcal{H}_I^{\text{fin}})$ -bimodules. Suppose f' is not an isomorphism, then there is a nonzero h in its kernel, i.e. $(f'(h))(t) = \theta'(th) = 0$ for all $t \in \mathcal{H}_n^{\text{fin}}$. Now write $h = \sum_{y \in D_I, l(y) \leq l(x)} T_y h_y$ for some $x \in D_I, h_y \in \mathcal{H}_I$. If $x = d$, we know that $f'(h)(1) = \theta(h) = h_d \neq 0$, so we can use downward induction on $l(x)$ to show $h = 0$. If $l(x) \leq l(d)$ we can find a transposition s such that $sx \in D_I$ and $l(sx) \geq l(x)$, so $T_s h = \sum_{y \in D_I, l(y) \leq l(sx)} T_y h'_y$ and $h'_{sx} = h_x \neq 0$, so by the inductive assumption $\theta'(\mathcal{H}_n^{\text{fin}} h) = \theta'(\mathcal{H}_n^{\text{fin}} T_s h) \neq 0$, whence f' is indeed an isomorphism and we're done. \square

Corollary 2.6. *For $M \in \mathcal{H}_I\text{-mod}^{\text{fd}}$, there is a natural isomorphism*

$$\text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, M) \cong \mathcal{H}_n \otimes_{\mathcal{H}_I} {}^d M$$

of \mathcal{H}_n -modules.

Proof. Analogous to [3], Corollary 2.12. \square

On \mathcal{H}_n , we can define an antiautomorphism τ defined on the generators as follows:

$$\begin{aligned} \tau : T_i &\mapsto T_i, \\ X_j &\mapsto X_j \end{aligned}$$

for all $i = 0, \dots, n-1, j = 0, \dots, n$. Since the relations given in Section 1 are all invariant with respect to reversal of the order of generators, this does indeed define an antiautomorphism.

As any antiautomorphism, τ can be used to define a left action of \mathcal{H}_n on the F -dual $M^* = \text{Hom}_F(M, F)$ of a module $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ via $hf(m) = f(\tau(h)m)$. Denote this module by M^τ . As τ leaves parabolic subalgebras of \mathcal{H}_n invariant, it can also be used to define a duality on finite dimensional \mathcal{H}_I -modules for any subset I of $\{0, \dots, n-1\}$. If we think of representations in terms of matrices, the τ -dual corresponds to taking the transposes of the representing matrices. Then we obtain another corollary of Lemma 2.5:

Corollary 2.7. *For $M \in \mathcal{H}_I\text{-mod}^{\text{fd}}$, there is a natural isomorphism*

$$(\text{ind}_I^n M)^\tau \cong \text{ind}_I^n ({}^d(M^\tau)).$$

Proof. Recall first that $I \cap dI = I$, so the induction on the right actually makes sense. Then the proof is analogous to [3], Lemma 2.13. \square

3. FORMAL CHARACTERS

In this section we investigate how far formal characters - mainly tuples of eigenvalues -, which uniquely determine irreducible modules for the affine Hecke algebra

of type A , lead in type B . In the following, we will make heavy use of the following well-known lemma.

Lemma 3.1. *For F -algebras A and B , the irreducibles in $A \otimes_F B\text{-mod}^{\text{fd}}$ are exactly the outer tensor products $M \boxtimes N$ of irreducible $M \in A\text{-mod}^{\text{fd}}$, $N \in B\text{-mod}^{\text{fd}}$. Further, if $M \boxtimes N \cong M' \boxtimes N'$, then $M \cong M'$ and $N \cong N'$.*

For $\underline{a} = (a_0, a_1, \dots, a_n) \in F^{n+1}$, the one-dimensional \mathcal{P}_n -module on which X_i acts as the scalar a_i for $0 \leq i \leq n$ will also be denoted by \underline{a} . If an eigenvalue occurs several times this will be indicated by an exponent in parentheses. So $(a_0, a^{(n)})$ is the one-dimensional \mathcal{P}_n -module on which X_0 acts as a_0 and all X_i for $i > 0$ act as a . Since \mathcal{P}_n is commutative and F is algebraically closed, the modules (a_0, a_1, \dots, a_n) exhaust all irreducibles in $\mathcal{P}_n\text{-mod}^{\text{fd}}$. This follows from the fact that commuting matrices can simultaneously be brought to upper triangular form.

For $M \in \mathcal{P}_n\text{-mod}^{\text{fd}}$ and any $\underline{a} \in F^{n+1}$, let $M[\underline{a}]$ be the largest submodule of M all of whose composition factors are isomorphic to \underline{a} , i.e.

$$M[\underline{a}] = \{m \in M \mid \exists k \in \mathbb{Z}_{\geq 0} \forall 0 \leq i \leq n : (X_i - a_i)^k m = 0\}$$

is the simultaneous generalized (a_0, a_1, \dots, a_n) -eigenspace for X_0, X_1, \dots, X_n .

Lemma 3.2. *For any $M \in \mathcal{P}_n\text{-mod}^{\text{fd}}$ we have $M = \bigoplus_{\underline{a} \in F^{n+1}} M[\underline{a}]$ as a \mathcal{P}_n -module.*

Proof. As an $F[X_i]$ -module we can decompose M into the direct sum of generalized eigenspaces since F is algebraically closed and we have a Jordan normal form. Since \mathcal{P}_n is commutative this decomposition is respected by all the other X_j and the assertion follows by induction. \square

We define the *formal character* of $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ by:

$$(3.1) \quad \text{ch } M := [\text{res}_{\mathcal{P}_n}^{\mathcal{H}_n} M] \in K(\mathcal{P}_n\text{-mod}^{\text{fd}}).$$

Exactness of the functor $\text{res}_{\mathcal{P}_n}^{\mathcal{H}_n}$ implies that ch induces a homomorphism

$$\text{ch} : K(\mathcal{H}_n\text{-mod}^{\text{fd}}) \rightarrow K(\mathcal{P}_n\text{-mod}^{\text{fd}})$$

between the corresponding Grothendieck groups. For $M \in \mathcal{H}_I\text{-mod}^{\text{fd}}$, the definition is modified in the obvious way. Note that if we expand

$$\text{ch } M = \sum_{\underline{a} \in F^{n+1}} r_{\underline{a}}[(a_0, a_1, \dots, a_n)]$$

in terms of the basis for $K(\mathcal{P}_n\text{-mod}^{\text{fd}})$ given by the irreducibles, the coefficient $r_{\underline{a}}$ is exactly the dimension of the generalized simultaneous \underline{a} -eigenspace $M[\underline{a}]$ of X_0, \dots, X_n .

We can explicitly compute formal characters of induced modules.

Lemma 3.3. *Let $\underline{a} = (a_0, a_1, \dots, a_n) \in F^{n+1}$. Then*

$$\text{ch ind}_{\mathcal{P}_n}^{\mathcal{H}_n} \underline{a} = \sum_{\substack{u \in S_n \\ \underline{\epsilon} \in \{1, -1\}^n}} [(b_0(u, \underline{\epsilon}), a_{u^{-1}(1)}^{\epsilon_1}, \dots, a_{u^{-1}(n)}^{\epsilon_n})]$$

where $b_0(u, \underline{\epsilon}) := a_0 \prod_{\substack{j \in \{1, \dots, n\} \\ \epsilon_j = -1}} a_{u^{-1}(j)}$.

Proof. This follows directly from the Mackey Theorem with $I = J = \emptyset$. \square

Lemma 3.4. (“Shuffle Lemma”)

Let $n = m + k$, and let $M \in \mathcal{H}_m\text{-mod}^{\text{fd}}$, $K \in \mathcal{H}_k^A\text{-mod}^{\text{fd}}$. Assume

$$\text{ch } M = \sum_{\underline{a} \in F^{m+1}} r_{\underline{a}}[(a_0, \dots, a_m)], \quad \text{ch } K = \sum_{\underline{b} \in F^k} t_{\underline{b}}[(b_1, \dots, b_k)].$$

Then

$$\text{ch ind}_{m,k}^n M \boxtimes K = \sum_{\underline{a} \in F^m} \sum_{\underline{b} \in F^k} r_{\underline{a}} t_{\underline{b}} \left(\sum_{\underline{c}} (c_0, c_1, \dots, c_n) \right),$$

where the last sum is over all $\underline{c} = (c_0, \dots, c_n) \in F^{n+1}$ such that (c_1, \dots, c_n) is obtained by shuffling (a_1, \dots, a_n) and $\underline{b}^\epsilon := (b_1^{\epsilon_1}, \dots, b_k^{\epsilon_k})$ for $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_k) \in \{1, -1\}^k$, which means there exist $1 \leq u_1 < \dots < u_m \leq n$ such that $(c_{u_1}, \dots, c_{u_m}) = (a_1, \dots, a_m)$, $(c_1, \dots, \hat{c}_{u_1}, \dots, \hat{c}_{u_m}, \dots, c_n) = \underline{b}^\epsilon$ and $c_0 = a_0 \prod_{\substack{j \in \{1, \dots, k\} \\ \epsilon_j = -1}} b_j$.

Proof. This also follows directly from the Mackey Theorem setting $I = \emptyset$ and $J = (m, k)$. \square

At this point, it is convenient to introduce the left coset representatives of the two maximal parabolic subgroups that will be most important in the following.

The set $D_{(n-1,1)}$ of distinguished left coset representatives of W_{n-1}^{fin} in W_n^{fin} consists of $1, s_j, s_{j,n-1}$ for $0 \leq j \leq n-1$ and $s_{j,0,n-1}$ for $1 \leq j \leq n-1$, see e.g. [5].

The set $D_{(0,n)}$ can easily be checked to consist of all elements of the form

$$s_{\underline{j}} := s_{j_r,0} s_{j_{r-1},0} \cdots s_{j_1,0}$$

for subsets $\underline{j} = (j_1 > j_2 > \dots > j_r) \subseteq \{0, \dots, n-1\}$.

To illustrate the way of computing formal characters using the Shuffle Lemma, we will now give two examples of induced modules from parabolic subalgebras of the types described above.

Example 3.5. First we take the irreducible module for $\mathcal{H}_{1,1}$ which is isomorphic to the outer tensor product of the one-dimensional \mathcal{H}_1 -module $L(a_0, p^2)$ on which T_0 acts as p , X_0 as a_0 and X_1 as p^2 and the one-dimensional $\mathcal{H}_1^A \cong F[X_2^{\pm 1}]$ -module (c) on which X_2 acts by multiplication with c . The set $D_{\emptyset, (1,1)}$, which is simply $D_{(1,1)}$ can be ordered as $(1, s_1, s_0 s_1, s_1 s_0 s_1)$ and the Shuffle Lemma yields

$$\begin{aligned} \text{ch ind}_{\mathcal{H}_{1,1}}^{\mathcal{H}_2} L(a_0, p^2) \boxtimes (c) &= [(a_0, p^2, c)] + [(a_0, c, p^2)] \\ &\quad + [(a_0 c, c^{-1}, p^2)] + [(a_0 c, p^2, c^{-1})]. \end{aligned}$$

As a second example, we consider $\text{ind}_{\mathcal{P}_0 \otimes \mathcal{H}_3^A}^{\mathcal{H}_3} (a_0) \boxtimes L^A(-q^{-2}, -1, -q^2)$, where $L^A(-q^{-2}, -1, -q^2)$ denotes the one-dimensional \mathcal{H}_3^A -module, on which T_1 and T_2 act by multiplication with q , and X_1, X_2, X_3 act as $-q^{-2}, -1, -q^2$ respectively. Ordering the set $D_{\emptyset, (0,3)} = D_{(0,3)}$ as

$$(1, s_0, s_1 s_0, s_2 s_1 s_0, s_0 s_1 s_0, s_0 s_2 s_1 s_0, s_1 s_0 s_2 s_1 s_0, s_0 s_1 s_0 s_2 s_1 s_0),$$

we obtain

$$\begin{aligned} \text{ch ind}_{\mathcal{P}_0 \otimes \mathcal{H}_3^A}^{\mathcal{H}_3} (a_0) \boxtimes L^A(-q^{-2}, -1, -q^2) &= [(a_0, -q^{-2}, -1, -q^2)] + [(-a_0 q^{-2}, -q^2, -1, -q^2)] \\ &\quad + [(-a_0 q^{-2}, -1, -q^2, -q^2)] + [(-a_0 q^{-2}, -1, -q^2, -q^2)] \\ &\quad + [(a_0 q^{-2}, -1, -q^2, -q^2)] + [(a_0 q^{-2}, -1, -q^2, -q^2)] \\ &\quad + [(a_0 q^{-2}, -q^2, -1, -q^2)] + [(-a_0, -q^{-2}, -1, -q^2)]. \end{aligned}$$

Recall that the center $Z(\mathcal{H})$ of \mathcal{H} consists of those Laurent polynomials f which are invariant under the action of W_n^{fin} . Given $\underline{a} \in F^{n+1}$, we associate the *central character*

$$\chi_{\underline{a}} : Z(\mathcal{H}_n) \rightarrow F, \quad f(X_0, \dots, X_n) \mapsto f(a_0, \dots, a_n),$$

so central characters are simply all algebra homomorphisms from $Z(\mathcal{H}_n)$ to F . Consider the left action of W_n^{fin} on F^{n+1} induced by the action on \mathcal{P}_n , i.e. $s_i(a_0, \dots, a_n) =$

$(a_0, \dots, a_{i+1}, a_i, \dots, a_n)$ for $1 \leq i \leq n-1$ and $s_0(a_0, \dots, a_n) = (a_0 a_1, a_1^{-1}, a_2, \dots, a_n)$. Writing $\underline{a} \sim \underline{b}$ if \underline{a} and \underline{b} lie in the same orbit with respect to this action, we get:

Lemma 3.6. *For $\underline{a}, \underline{b} \in F^{n+1}$, $\chi_{\underline{a}} = \chi_{\underline{b}}$ if and only if $\underline{a} \sim \underline{b}$.*

Thus the central characters of \mathcal{H}_n are actually labeled by the set F^{n+1}/\sim of W_n^{fin} -orbits on F_{n+1} and we set, for $\gamma \in F^{n+1}/\sim$

$$\chi_\gamma := \chi_{\underline{a}}$$

for any $\underline{a} \in \gamma$.

Accordingly, for $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ we define

$$M[\gamma] = \{v \in M \mid (z - \chi_\gamma(z))^k v = 0 \text{ for all } z \in Z(\mathcal{H}_n) \text{ and } k \gg 0\}.$$

It is important that this is an \mathcal{H}_n -submodule of M . Indeed, $(z - \chi_\gamma(z))^k v = 0$ implies $(z - \chi_\gamma(z))^k h v = h(z - \chi_\gamma(z))^k v = 0$ for all $h \in \mathcal{H}_n$. Now, for $\underline{a} \in F^{n+1}$ with $\underline{a} \in \gamma$, $Z(\mathcal{H}_n)$ acts on the \mathcal{P}_n -submodule $M[\underline{a}]$ via the central character χ_γ . Hence

$$M[\gamma] = \bigoplus_{\underline{a} \in \gamma} M[\underline{a}],$$

as a \mathcal{P}_n -module. So, recalling Lemma 3.2, we see:

Lemma 3.7. *Any $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ decomposes as*

$$M = \bigoplus_{\gamma \in F^{n+1}/\sim} M[\gamma]$$

as an \mathcal{H}_n -module.

Thus the χ_γ , $\gamma \in F^{n+1}/\sim$, exhaust the possible central characters that can arise in a finite dimensional \mathcal{H}_n -module, while Lemma 3.3 shows that every such central character does arise in some finite dimensional \mathcal{H}_n -module.

For $\gamma \in F^{n+1}/\sim$, we define $\mathcal{H}_n[\gamma]\text{-mod}^{\text{fd}}$ to be the full subcategory of $\mathcal{H}_n\text{-mod}^{\text{fd}}$ consisting of all modules M with $M[\gamma] = M$. Lemma 3.7 in fact yields an equivalence of categories

$$(3.2) \quad \mathcal{H}_n\text{-mod}^{\text{fd}} \cong \bigoplus_{\gamma \in F^{n+1}/\sim} \mathcal{H}_n[\gamma]\text{-mod}^{\text{fd}}.$$

In the following, we will call $\mathcal{H}_n[\gamma]\text{-mod}^{\text{fd}}$ the block of $\mathcal{H}_n\text{-mod}^{\text{fd}}$ corresponding to γ .

Observe that none of this actually uses which affine Hecke algebra we work with, thus the analogous statements hold for any parabolic subalgebra of \mathcal{H}_n as well as of \mathcal{H}_n^R , in particular we have the same notions of formal characters, central characters and blocks for \mathcal{H}_n^A .

We now need a type A result for a special module that will be important in the following section. It is due to Kato [9], but for convenience, we include a proof. Denote by $L^A(a^{(n)}) := \text{ind}_{\mathcal{R}_n}^{\mathcal{H}_n^A}(a^{(n)})$ the principal series module in type A.

Lemma 3.8. *Let $a \in F$ and $I = (0, m_1, \dots, m_r) \subseteq \{0, \dots, n-1\}$.*

- (i) $L^A(a^{(n)})$ is irreducible, and it is the only irreducible module in its block.
- (ii) All composition factors of $\text{res}_{\mathcal{H}_I^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ are isomorphic to

$$L^A(a^{(m_1)}) \boxtimes \dots \boxtimes L^A(a^{(m_r)}),$$

and $\text{soc res}_{\mathcal{H}_I^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ is irreducible.

- (iii) $\text{soc res}_{\mathcal{H}_{n-1}^A}^{\mathcal{H}_n^A} L^A(a^{(n)}) \cong L^A(a^{(n-1)})$.

- (iv) The size of any Jordan block of X_n on $L^A(a^{(n)})$ is n .

Proof. We first show that the $a^{(n-1)}$ -eigenspace for X_1, \dots, X_{n-1} is contained in $1 \otimes (a^{(n)})$, which is, of course, also contained in the a -eigenspace of X_n . This is certainly true for $n = 1$, since in this case $L^A(a) = (a)$ as $\mathcal{H}_1^A = \mathcal{R}_1$. So assume that the statement holds for $n - 1$, i.e. the $a^{(n-2)}$ -eigenspace for X_1, \dots, X_{n-2} in $L^A(a^{(n-1)})$ is contained in $1 \otimes (a^{(n-1)})$. Let $u \in L^A(a^{(n)})$ be an $a^{(n-1)}$ -eigenvector for X_1, \dots, X_{n-1} . As $L^A(a^{(n)}) \cong \text{ind}_{\mathcal{H}_{n-1,1}^A}^{\mathcal{H}_n^A} L^A(a^{(n-1)}) \boxtimes (a)$, u can be written as

$$u = \sum_{j=1}^{n-1} T_{j,n-1} \otimes u_j + 1 \otimes u_0$$

with $u_j \in L^A(a^{(n-1)}) \boxtimes (a)$ for $0 \leq j \leq n - 1$. Now suppose l is minimal such that $u_l \neq 0$, then, if $l \leq n - 1$

$$\begin{aligned} & (X_{l+1} - a)(T_{l,n-1} \otimes u_l + T_{l+1,n-1} \otimes u_{l+1}) \\ &= T_{l,n-1}(X_l - a) \otimes u_l + (q - q^{-1})T_{l+1,n-1}X_n \otimes u_l \\ & \quad + T_{l+1,n-1}(X_n - a) \otimes u_{l+1} \\ & \quad + \text{terms in } T_{j,n-1} \otimes L^A(a^{(n-1)}) \boxtimes (a) \text{ for } j \geq l + 2 \text{ and } 1 \otimes L^A(a^{(n-1)}). \end{aligned}$$

Since X_n acts as a on $L^A(a^{(n-1)}) \boxtimes (a)$, we see from the coefficient of $T_{l+1,n-1}$ that u_l has to be zero, a contradiction. If $l = n - 1$, then for $k \leq n - 2$

$$\begin{aligned} & (X_k - a)(T_{n-1} \otimes u_{n-1} + 1 \otimes u_0) \\ &= T_{n-1}(X_k - a) \otimes u_{n-1} + 1 \otimes (X_k - a)u_0, \end{aligned}$$

whence, u_{n-1}, u_0 have to be contained in $1 \otimes (a^{(n-1)}) \boxtimes (a) \subseteq L^A(a^{(n-1)}) \boxtimes (a)$ by the inductive hypothesis. But

$$\begin{aligned} & (X_{n-1} - a)(T_{n-1} \otimes u_{n-1} + 1 \otimes u_0) \\ &= T_{n-1}(X_n - a) \otimes u_{n-1} \\ & \quad - (q - q^{-1})1 \otimes X_n u_{n-1} + 1 \otimes (X_{n-1} - a)u_0, \end{aligned}$$

requiring $(X_{n-1} - a)u_0 = (q - q^{-1})1 \otimes X_n u_{n-1}$, but $(X_{n-1} - a)u_0 = 0$ as we have just seen. Hence, l does not exist, so the $a^{(n)}$ -eigenspace is contained in $1 \otimes L^A(a^{(n-1)}) \boxtimes (a)$ and therefore, by induction, in $1 \otimes (a^{(n)})$.

Now any nontrivial submodule of $L^A(a^{(n)})$ has to contain a simultaneous eigenvector for X_1, \dots, X_n which can only be an $a^{(n)}$ -eigenvector as the formal character of $L^A(a^{(n)})$ is $n![(a^{(n)})]$. Therefore any nontrivial submodule contains $1 \otimes (a^{(n)})$, generating the whole of $L^A(a^{(n)})$. This proves (i).

The assertion that all composition factors of $\text{res}_{\mathcal{H}_t^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ are isomorphic to $L^A(a^{(m_1)}) \boxtimes \dots \boxtimes L^A(a^{(m_r)})$ follows immediately from the Mackey Theorem and the irreducibility of the latter module by (i) and Lemma 3.1. Therefore, the socle of $\text{res}_{\mathcal{H}_t^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ consists of a number of copies of $L^A(a^{(m_1)}) \boxtimes \dots \boxtimes L^A(a^{(m_r)})$. But any constituent of the socle contains a simultaneous eigenvector for \mathcal{R}_n , but there is only one such, up to a scalar multiple, hence the socle is irreducible, completing the proof of (ii).

By (ii) the socle of $\text{res}_{\mathcal{H}_{n-1,1}^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ is isomorphic to $L^A(a^{(n-1)}) \boxtimes (a)$, hence the socle of $\text{res}_{\mathcal{H}_{n-1}^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ certainly contains a copy of $L^A(a^{(n-1)})$. But again, every constituent of $\text{soc res}_{\mathcal{H}_{n-1}^A}^{\mathcal{H}_n^A} L^A(a^{(n)})$ contains a simultaneous eigenvector of \mathcal{R}_{n-1} , which also is contained in $1 \otimes (a^{(n)})$, hence $\text{soc res}_{\mathcal{H}_{n-1}^A}^{\mathcal{H}_n^A} L^A(a^{(n)}) \cong L^A(a^{(n-1)})$.

For the proof of (iv), assume inductively that the size of any Jordan block of X_{n-1} on $L^A(a^{(n-1)})$ is $n-1$ which we can do since the statement certainly holds for the \mathcal{H}_1^A -module (a) . Hence there are $(n-2)!$ elements in $L^A(a^{(n-1)})$, each generating a Jordan block of size $n-1$ for X_{n-1} and similarly for $L^A(a^{(n-1)}) \boxtimes (a)$. Now for such an element v and any $1 \leq l \leq n-1$

$$(X_n - a)T_{l,n-1} \otimes v = T_{l,n-1}(X_{n-1} - a) \otimes v + (q - q^{-1})T_{l,n-2}(X_n) \otimes v$$

whence

$$(X_n - a)^{n-1}T_{l,n-1} \otimes v = (q - q^{-1})T_{l,n-2} \otimes (X_n)(X_{n-1} - a)^{n-2}v$$

which is nonzero but annihilated by $(X_n - a)$. So we have $(n-1)(n-2)! = (n-1)!$ Jordan blocks of size n for X_n , exhausting $L^A(a^{(n)})$. \square

In type B, we need certain restrictions on the eigenvalue a to prove a similar statement.

Lemma 3.9. *Let $a \in F \setminus \{\pm 1\}$. Then*

- (i) $\text{ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(a_0, a^{(n)})$ has an irreducible cosocle and
- (ii) this cosocle, denoted by $L(a_0, a^{(n)})$, is the only irreducible module in $\mathcal{H}_n\text{-mod}^{\text{fd}}$ containing $[(a_0, a^{(n)})]$ as a formal character.

Proof. Note that

$$\begin{aligned} \text{ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(a_0, a^{(n)}) &\cong \text{ind}_{\mathcal{P}_0 \otimes \mathcal{H}_n^A}^{\mathcal{H}_n} \text{ind}_{\mathcal{P}_n}^{\mathcal{P}_0 \otimes \mathcal{H}_n^A}(a_0, a^{(n)}) \\ &\cong \text{ind}_{\mathcal{P}_0 \otimes \mathcal{H}_n^A}^{\mathcal{H}_n}(a_0) \boxtimes L^A(a^{(n)}). \end{aligned}$$

Now, applying the Shuffle Lemma, we see that in fact, the only formal characters of the form $[(a_0, a^{(n)})]$ in $\text{ind}_{\mathcal{P}_0 \otimes \mathcal{H}_n^A}^{\mathcal{H}_n}(a_0) \boxtimes L^A(a^{(n)})$ stem from the coset representative 1, from which it follows that such formal character values can occur only in the cosocle of the induced module. Knowing this, (ii) will follow as soon as we have established (i). But from Frobenius reciprocity (2.2), we have

$$\begin{aligned} &\text{Hom}_{\mathcal{H}_n}(\text{ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(a_0, a^{(n)}), \text{cosoc } \text{ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(a_0, a^{(n)})) \\ &\cong \text{Hom}_{\mathcal{P}_0 \otimes \mathcal{H}_n^A}((a_0) \boxtimes L^A(a^{(n)}), \text{res}_{\mathcal{P}_0 \otimes \mathcal{H}_n^A}^{\mathcal{H}_n} \text{cosoc } \text{ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(a_0, a^{(n)})) \\ &= F, \end{aligned}$$

proving (i). \square

4. “KASHIWARA OPERATORS”

In this section we will, in analogy to the type A situation, define maps between the sets of isomorphism classes of irreducibles in $\mathcal{H}_n\text{-mod}^{\text{fd}}$ and $\mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$.

Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ and $a \in F$. Define $\Delta_a M$ to be the generalized a -eigenspace of X_n in M , i.e.

$$\Delta_a M = \bigoplus_{\underline{a} \in F^{n+1}, a_n = a} M[\underline{a}].$$

As X_n is central in the parabolic subalgebra $\mathcal{H}_{n-1,1} \cong \mathcal{H}_{n-1} \otimes F[X_n^{\pm 1}]$ of \mathcal{H}_n , $\Delta_a M$ is an $\mathcal{H}_{n-1,1}$ -submodule of M . Thus Δ_a defines a functor

$$\Delta_a : \mathcal{H}_n\text{-mod}^{\text{fd}} \rightarrow \mathcal{H}_{n-1,1}\text{-mod}^{\text{fd}},$$

which, on morphisms, is simply restriction. This functor is exact as the composite of restriction to a subalgebra and then taking a direct summand. Analogously, for $m \geq 0$, $\Delta_{a^{(m)}}$ denotes the functor $\mathcal{H}_n\text{-mod}^{\text{fd}} \rightarrow \mathcal{H}_{n-m,m}\text{-mod}^{\text{fd}}$ that maps M to simultaneous generalized a -eigenspace of X_{n-m+1}, \dots, X_n .

By Lemmas 3.1 and 3.8, $\Delta_{a^{(m)}}M$ is the largest submodule of $\text{res}_{n-m,m}^n M$ all of whose composition factors are of the form $N \boxtimes L^A(a^{(m)})$ for some irreducible $N \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}}$ and is indeed a direct summand of $\text{res}_{n-m,m}^n M$.

Lemma 4.1. *For $N \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}}$, $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$, there is a functorial isomorphism*

$$\text{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L^A(a^{(m)}), \Delta_{a^{(m)}}M) \cong \text{Hom}_{\mathcal{H}_n}(\text{ind}_{n-m,m}^n N \boxtimes L^A(a^{(m)}), M).$$

Proof. By Lemma 3.1, all composition factors of $\text{res}_{n-m,m}^n M$ are of the form $K \boxtimes L$ for irreducible $K \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}}$ and $L \in \mathcal{H}_m^A\text{-mod}^{\text{fd}}$. An injection of the irreducible $N \boxtimes L^A(a^{(m)})$ into $\text{res}_{n-m,m}^n M$ can only map onto a submodule that is isomorphic to $N \boxtimes L^A(a^{(m)})$ and all composition factors with this property are contained in $\Delta_{a^{(m)}}M$. Since $\Delta_{a^{(m)}}M$ is a direct summand of $\text{res}_{n-m,m}^n M$, the assertion follows from Frobenius reciprocity (2.2). \square

The following is immediate from the definition:

Lemma 4.2. *Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ with*

$$\text{ch } M = \sum_{\underline{a} \in F^{n+1}} r_{\underline{a}}[(a_0, \dots, a_n)].$$

Then we have

$$\text{ch } \Delta_{a^{(m)}}M = \sum_{\underline{b}} r_{\underline{b}}[(b_0, \dots, b_n)],$$

summing over all $\underline{b} \in F^{n+1}$ with $b_{n-m+1} = \dots = b_n = a$.

Now for $a \in F$ and $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$, define

$$\epsilon_a(M) = \max\{m \geq 0 \mid \Delta_{a^{(m)}}M \neq 0\}.$$

By Lemma 4.2, $\epsilon_a(M)$ is simply the length of the ‘longest a -tail’ in $\text{ch } M$.

Lemma 4.3. *Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ be irreducible, $a \in F$, $\epsilon = \epsilon_a(M)$. If $N \boxtimes L^A(a^{(m)})$ is an irreducible submodule of $\Delta_{a^{(m)}}M$ for some $0 < m \leq \epsilon$ and some irreducible $N \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}}$, then $\epsilon_a(N) = \epsilon - m$.*

Proof. Analogous to [3], Lemma 5.2. \square

Lemma 4.4. *Let $m \geq 0$, $a \in F \setminus \{\pm 1\}$ and $N \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}}$ be irreducible with $\epsilon_a(N) = 0$. Set $M = \text{ind}_{n-m,m}^n N \boxtimes L^A(a^{(m)})$. Then*

- (i) $\Delta_{a^{(m)}}M \cong N \boxtimes L^A(a^{(m)})$;
- (ii) $\text{cosoc } M$ is irreducible with $\epsilon_a(\text{cosoc } M) = m$;
- (iii) All other composition factors L of M have $\epsilon_a(L) < m$.

Proof. Analogous to [3], Lemma 5.3. \square

Lemma 4.5. *Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ be irreducible, $a \in F \setminus \{\pm 1\}$, $\epsilon = \epsilon_a(M)$ and $0 \leq m \leq \epsilon$. Then*

- (i) $\Delta_{a^{(\epsilon)}}M \cong N \boxtimes L^A(a^{(\epsilon)})$ for some irreducible $N \in \mathcal{H}_{n-\epsilon}\text{-mod}^{\text{fd}}$ with $\epsilon_a(N) = 0$,
- (ii) $\text{soc } \Delta_{a^{(m)}}M \cong L \boxtimes L^A(a^{(m)})$ for some irreducible $L \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}}$ with $\epsilon_a(L) = \epsilon_a(M) - m$.

In particular, if ± 1 do not occur as eigenvalues of X_n on M , $\text{soc } \text{res}_{n-1,1}^n M$ is multiplicity-free.

Proof. (i) Pick an irreducible submodule $N \boxtimes L^A(a^{(\epsilon)})$ of $\Delta_{a^{(\epsilon)}} M$, cf. remark before Lemma 4.1. Then $\epsilon_a(N) = 0$ by Lemma 4.3. By Lemma 4.1

$$\mathrm{Hom}_{\mathcal{H}_{n-\epsilon,\epsilon}}(N \boxtimes L^A(a^{(\epsilon)}), \Delta_{a^{(\epsilon)}} M) \cong \mathrm{Hom}_{\mathcal{H}_n}(\mathrm{ind}_{n-\epsilon,\epsilon}^n N \boxtimes L^A(a^{(\epsilon)}), M) \neq 0,$$

thus M , being irreducible, is a quotient of $\mathrm{ind}_{n-\epsilon,\epsilon}^n N \boxtimes L^A(a^{(\epsilon)})$. Hence, $\Delta_{a^{(\epsilon)}} M$ is a quotient of $\Delta_{a^{(\epsilon)}} \mathrm{ind}_{n-\epsilon,\epsilon}^n N \boxtimes L^A(a^{(\epsilon)})$. But, by Lemma 4.4(i), this is irreducible and isomorphic to $N \boxtimes L^A(a^{(\epsilon)})$, proving (i).

(ii) For every constituent $L \boxtimes L^A(a^{(m)})$ of $\mathrm{soc} \Delta_{a^{(m)}} M$, Lemma 4.3 tells us that $\epsilon_a(L) = \epsilon - m$, hence $\Delta_{a^{(\epsilon-m)}} L \boxtimes L^A(a^{(m)})$ is a non-trivial submodule of $\mathrm{res}_{n-\epsilon,\epsilon-m,m}^{n-\epsilon,\epsilon} \Delta_{a^{(\epsilon)}} M$. By (i), $\Delta_{a^{(\epsilon)}} M$ is irreducible of the form $N \boxtimes L^A(a^{(\epsilon)})$, so Lemma 3.8(ii) implies that

$$\mathrm{soc} \mathrm{res}_{n-\epsilon,\epsilon-m,m}^{n-\epsilon,\epsilon} \Delta_{a^{(\epsilon)}} M \cong N \boxtimes L^A(a^{(\epsilon-m)}) \boxtimes L^A(a^{(m)}).$$

Therefore there can only be one such constituent in $\mathrm{soc} \Delta_{a^{(m)}} M$ and we're done. \square

Defining

$$(4.1) \quad e_a := \mathrm{res}_{n-1}^{n-1,1} \circ \Delta_a : \mathcal{H}_n\text{-mod}^{\mathrm{fd}} \rightarrow \mathcal{H}_{n-1}\text{-mod}^{\mathrm{fd}}$$

and analogously

$$f_a = \mathrm{ind}_{n-1,1}^n - \boxtimes(a) : \mathcal{H}_{n-1}\text{-mod}^{\mathrm{fd}} \rightarrow \mathcal{H}_n\text{-mod}^{\mathrm{fd}}$$

we obtain the following corollary.

Corollary 4.6. *For $a \in F \setminus \{\pm 1\}$ and an irreducible $M \in \mathcal{H}_n\text{-mod}^{\mathrm{fd}}$ with $\epsilon_a(M) > 0$, $\mathrm{soc} e_a M$ is irreducible, and $\epsilon_a(\mathrm{soc} e_a M) = \epsilon_a(M) - 1$.*

Proof. Choose a constituent L of $\mathrm{soc} e_a M$. The central element $Z := X_0^2 X_1 \dots X_n$ of \mathcal{H}_n acts as a scalar on the whole of M by Schur's Lemma, and similarly the central element $Z' := X_0^2 X_1 \dots X_{n-1}$ of \mathcal{H}_{n-1} acts as a scalar on L . Hence $X_n = Z'^{-1} Z$ acts on L as a scalar, too. The scalar must be a , so L contributes a constituent $L \boxtimes (a)$ to $\mathrm{soc} \Delta_a M$. By Lemma 4.5 (ii) this is irreducible and satisfies $\epsilon_a(\mathrm{soc} \Delta_a M) = \epsilon_a(M) - 1$, so $\mathrm{soc} e_a M$ must also be irreducible and isomorphic to L and $\epsilon_a(\mathrm{soc} e_a M) = \epsilon_a(M) - 1$. \square

The following lemma provides a recipe for an inductive construction of irreducible modules in $\mathcal{H}_n\text{-mod}^{\mathrm{fd}}$ from irreducibles in $\mathcal{H}_{n-1}\text{-mod}^{\mathrm{fd}}$.

Lemma 4.7. *Let $m \geq 0$, $a \in F \setminus \{\pm 1\}$, let $N \in \mathcal{H}_n\text{-mod}^{\mathrm{fd}}$ be irreducible and set $M = \mathrm{ind}_{n,m}^{n+m}(N \boxtimes L^A(a^{(m)}))$. Then, $\mathrm{cosoc} M$ is irreducible with $\epsilon_a(\mathrm{cosoc} M) = \epsilon_a(N) + m$, and all other composition factors L of M have $\epsilon_a(L) < \epsilon_a(N) + m$.*

Proof. Analogous to [3], Lemma 5.5. \square

We can now define the desired Kashiwara type operators. Let M be an irreducible module in $\mathcal{H}_n\text{-mod}^{\mathrm{fd}}$. Define

$$(4.2) \quad \tilde{e}_a M := \mathrm{soc} e_a M, \quad \tilde{f}_a M := \mathrm{cosoc} f_a M = \mathrm{cosoc} \mathrm{ind}_{n,1}^{n+1} M \boxtimes (a),$$

For $a \neq \pm 1$, $\tilde{f}_a M$ is irreducible by Lemma 4.7 and $\tilde{e}_a M$ is irreducible or 0 by Corollary 4.6, hence the functors induce maps between the set of isomorphism classes of irreducibles in $\mathcal{H}_n\text{-mod}^{\mathrm{fd}}$ and $\mathcal{H}_{n-1}\text{-mod}^{\mathrm{fd}}$. Observe that Corollary 4.6 implies

$$(4.3) \quad \epsilon_a(M) = \max\{m \geq 0 \mid \tilde{e}_a^m M \neq 0\}$$

and, by Lemma 4.7, we have

$$(4.4) \quad \epsilon_a(\tilde{f}_a M) = \epsilon_a(M) + 1.$$

Lemma 4.8. *Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ be irreducible, $a \in F \setminus \{\pm 1\}$ and $m \geq 0$.*

- (i) $\text{soc } \Delta_{a^{(m)}} M \cong (\tilde{e}_a^m M) \boxtimes L^A(a^{(m)})$.
- (ii) $\text{cosoc } \text{ind}_{n,m}^{n+m} M \boxtimes L^A(a^{(m)}) \cong \tilde{f}_a^m M$.

Proof. Analogous to [3], Lemma 5.9. \square

Lemma 4.9. *Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ and $N \in \mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$ be irreducible modules and $a \in F \setminus \{\pm 1\}$. Then, $\tilde{e}_a M \cong N$ if and only if $\tilde{f}_a N \cong M$.*

Proof. Analogous to [3], Lemma 5.10. \square

We immediately get the following corollary:

Corollary 4.10. *Let $M, N \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ be irreducible and $a \in F \setminus \{\pm 1\}$. Then*

- (i) $\tilde{e}_a \tilde{f}_a M \cong M$ and, if $\epsilon_a(M) > 0$, $\tilde{f}_a \tilde{e}_a M \cong M$;
- (ii) $\tilde{f}_a M \cong \tilde{f}_a N$ if and only if $M \cong N$ and, if $\epsilon_a(M), \epsilon_a(N) > 0$, $\tilde{e}_a M \cong \tilde{e}_a N$ if and only if $M \cong N$.

We define the graph Γ as the graph whose vertices correspond to isomorphism classes of irreducible modules in $\bigoplus_{n \geq 0} \mathcal{H}_n\text{-mod}^{\text{fd}}$, where there is an edge $[N] \xrightarrow{a} [M]$

if and only if $M \cong \tilde{f}_a N$. If an irreducible $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ is isomorphic to $\tilde{f}_{a_n} \cdots \tilde{f}_{a_1}(a_0)$, we write $M \cong L(a_0, a_1, \dots, a_n)$. Note that these definitions make sense even in the case where $a \in \{\pm 1\}$. Even though in this case \tilde{f}_a does not necessarily produce irreducible modules, as we will see in Section 5, we just don't draw any edges from N if $\text{cosoc } \text{ind}_{n-1,1}^n N \boxtimes (a)$ is not irreducible. This has the drawback that not every vertex in Γ is connected to a module for \mathcal{H}_0 , but it enables us to use the (by this definition unique) labeling $L(\underline{a})$ in the cases where 1 or -1 occur in \underline{a} , but the corresponding cosocle of the induced modules are irreducible.

We will denote the full subcategory of $\mathcal{H}_n\text{-mod}^{\text{fd}}$ consisting of those modules on which $(X_i \pm 1)$ acts invertibly for all $1 \leq i \leq n$ by $\text{Rep}_{\neq \pm 1} \mathcal{H}_n$. Then we obtain the following:

Theorem 4.11. *The map $\text{ch} : K(\text{Rep}_{\neq \pm 1} \mathcal{H}_n) \rightarrow K(\mathcal{P}_n\text{-mod}^{\text{fd}})$ is injective.*

Proof. Analogous to [3], Theorem 5.12. \square

Corollary 4.12. *If L is an irreducible module in $\text{Rep}_{\neq \pm 1} \mathcal{H}_n$, then $L \cong L^\tau$.*

Proof. Since $\tau(X_i) = X_i$, τ leaves characters invariant. Hence it leaves irreducibles invariant since they are determined up to isomorphism by their character according to the theorem. \square

We now give three interpretations of the functions ϵ_a .

Theorem 4.13. *Let $a \in F \setminus \{\pm 1\}$ and M be an irreducible module in $\mathcal{H}_n\text{-mod}^{\text{fd}}$. Then*

- (i) $[e_a M] = \epsilon_a(M)[\tilde{e}_a M] + \sum c_r[N_r]$ where the N_r are irreducible modules with $\epsilon_a(N_r) < \epsilon_a(\tilde{e}_a M)$;
- (ii) $\epsilon_a(M)$ is the maximal size of a Jordan block of X_n on M with eigenvalue a ;
- (iii) The algebra $\text{End}_{\mathcal{H}_{n-1}}(e_a M)$ is isomorphic to the algebra of truncated polynomials $F[x]/(x^{\epsilon_a(M)})$.

Proof. Let $\epsilon = \epsilon_a(M)$ and $N = \tilde{e}_a^\epsilon M$.

- (i) By Lemma 4.5 and Frobenius reciprocity, there is a short exact sequence

$$0 \longrightarrow K \longrightarrow \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)}) \longrightarrow M \longrightarrow 0,$$

where all composition factors L of K have $\epsilon_a(L) < \epsilon$ by Lemma 4.4(iii). Applying the exact functor Δ_a , we obtain the exact sequence

$$0 \longrightarrow \Delta_a K \longrightarrow \Delta_a \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)}) \longrightarrow \Delta_a M \longrightarrow 0.$$

By the Mackey Theorem

$$\begin{aligned} [\text{res}_{n-1,1}^n \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)})] &= [\text{ind}_{n-\epsilon, \epsilon-1,1}^{n-1,1} N \boxtimes \text{res}_{\mathcal{H}_{\epsilon-1,1}^A}^{\mathcal{H}_{\epsilon}^A} L^A(a^{(\epsilon)})] \\ &\quad + [\text{ind}_{n-\epsilon-1, \epsilon,1}^{n-1,1} \text{res}_{n-\epsilon-1,1}^{n-\epsilon} (N \boxtimes L^A(a^{(\epsilon)}))] \\ &\quad + [\text{ind}_{n-\epsilon, \epsilon-1,1}^{n-1,1} \text{res}_{\mathcal{H}_{1, \epsilon-1}^A}^{s_{n-1,0,n-\epsilon}} (N \boxtimes \text{res}_{\mathcal{H}_{\epsilon-1,1}^A}^{\mathcal{H}_{\epsilon}^A} L^A(a^{(\epsilon)}))] \end{aligned}$$

The third subquotient does not contribute to $\Delta_a \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)})$ as its formal character ends on a^{-1} . Similarly, the direct summands of $\text{res}_{\mathcal{H}_{\epsilon-1,1}^A}^{\mathcal{H}_{\epsilon}^A} L^A(a^{(\epsilon)})$ other than $\Delta_a L^A(a^{(\epsilon)})$ and the direct summands of $\text{res}_{n-\epsilon-1,1}^{n-\epsilon} N \boxtimes L^A(a^{(\epsilon)})$ other than $\Delta_a N \boxtimes L^A(a^{(\epsilon)})$ cannot contribute to $\Delta_a \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)})$. As $\Delta_a N = 0$, we obtain

$$\Delta_a \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)}) \cong \text{ind}_{n-\epsilon, \epsilon-1,1}^{n-1,1} N \boxtimes \Delta_a L^A(a^{(\epsilon)}).$$

By considering characters, we see that

$$[\Delta_a L^A(a^{(\epsilon)})] = \epsilon [L^A(a^{(\epsilon-1)}) \boxtimes (a)],$$

hence

$$[\Delta_a \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)})] = \epsilon [\text{ind}_{n-\epsilon, \epsilon-1,1}^{n-1,1} N \boxtimes L^A(a^{(\epsilon-1)}) \boxtimes (a)].$$

By Lemma 4.8(ii), the cosocle of $\text{ind}_{n-\epsilon, \epsilon-1,1}^{n-1,1} N \boxtimes L^A(a^{(\epsilon-1)}) \boxtimes (a)$ is $(\tilde{f}_a^{(\epsilon-1)} N) \boxtimes (a)$ which is the same as $(\tilde{e}_a M) \boxtimes (a)$, and all other composition factors of this module are of the form $L \boxtimes (a)$ with $\epsilon_a(L) < \epsilon - 1$ by Lemma 4.4. Moreover, all composition factors of $\Delta_a K$ are of the form $L \boxtimes (a)$ with $\epsilon_a(L) < \epsilon - 1$. So we have now seen that

$$[\Delta_a M] = \epsilon [\tilde{e}_a M \boxtimes (a)] + \sum c_r [N_r \boxtimes (a)]$$

for irreducibles N_r with $\epsilon_a(N_r) < \epsilon_a(\tilde{e}_a M)$, which implies (i).

The proofs of (ii) and (iii) are analogous to [11], Theorem 5.5.1. \square

An interesting consequence is the following.

Corollary 4.14. *Let $M, N \in \text{Rep}_{\neq \pm 1} \mathcal{H}_n$ be irreducible modules with $M \not\cong N$. Then, for $a \in F \setminus \{\pm 1\}$, we have $\text{Hom}_{\mathcal{H}_{n-1}}(e_a M, e_a N) = 0$.*

Proof. Analogous to [3], Corollary 6.12 or [11], Corollary 5.5.2. \square

5. DISCUSSION OF EIGENVALUES 1 AND -1

We now investigate the cases when some of the X_i have eigenvalues 1 and -1 . The usual formal character argument used in section 4 does not fully apply here, since, when we compute formal characters of induced modules $f_a N$, the term coming from the longest coset representative in the Shuffle Lemma, ends on a^{-1} which for $a = \pm 1$ is the same as a . Thus we obtain higher multiplicities of $N \boxtimes (a)$ in $\Delta_a f_a N$. First, we consider eigenspaces with a maximal number of a s at the end.

Lemma 5.1. *Let $N \in \mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$ be irreducible and $a = \pm 1$. Let $\epsilon_a(N) = r - 1 \geq 0$. Set $M := \text{ind}_{n-1,1}^n N \boxtimes (a)$. Let $(a_0, \underline{b}, a^{(r-1)})$ be a tuple of eigenvalues on N . Then any $(a_0, \underline{b}, a^{(r)})$ -eigenvector in M is either of the form*

$$1 \otimes v$$

or

$$T_{n-r,0,n-1} \otimes v + \sum_{j=0}^{n-r-1} T_{j,0,n-1} \otimes u_j + \sum_{j=1}^{n-1} T_{j,n-1} \otimes w_j + 1 \otimes u_0$$

where $v = \tilde{v} \boxtimes c$ for an $(a_0, \underline{b}, a^{(r-1)})$ - or $(a_0 a, \underline{b}, a^{(r-1)})$ -eigenvector \tilde{v} in N , a generator c of the one-dimensional module (a) and some u_j, w_j, u_0 in $N \boxtimes (a)$.

In other words, it has leading term 1 or $T_{n-r,0,n-1}$.

In particular, the $(a_0, \underline{b}, a^{(r)})$ -eigenspace in M has at most twice the dimension of that in $N \boxtimes (a)$.

Proof. Directly from the action of the Weyl group on the lattice, we see that the leading term has to be of the form $T_{l,0,n-1}$ for $l \geq n-r$ or $T_{l,n-1}$ for $l \geq n-r+1$ for there to be an a -tail of length r .

For leading term $T_{l,0,n-1}$ with $l > n-r$

$$\begin{aligned} & (X_l - a)(T_{l,0,n-1} \otimes u_l + T_{l-1,0,n-1} \otimes u_{l-1}) \\ &= T_{l,0,n-1}(X_l - a) \otimes u_l - (q - q^{-1})T_{l-1,0,n-1}X_l \otimes u_l \\ & \quad + T_{l-1,0,n-1}(X_n^{-1} - a) \otimes u_{l-1} + \text{lower terms} \end{aligned}$$

Since $X_n^{-1} - a$ acts as 0 on the whole of $N \boxtimes (a)$ and since none of the lower terms can contribute to the coefficient of $T_{l-1,0,n-1}$, this shows that $u_l = 0$ which contradicts the assumption that $T_{l,0,n-1}$ is the leading term. Thus the leading term can only be $T_{n-r,0,n-1}$ in this case.

Equally, if the leading term is $T_{l,n-1}$ for $l \geq n-r+1$

$$\begin{aligned} & (X_{l+1} - a)(T_{l,n-1} \otimes w_l + T_{l+1,n-1} \otimes w_{l+1}) \\ &= T_{l,n-1}(X_l - a) \otimes w_l + (q - q^{-1})T_{l+1,n-1}X_n \otimes w_l \\ & \quad + T_{l+1,n-1}(X_n - a) \otimes w_{l+1} + \text{lower terms} \end{aligned}$$

By the same argument as above, $w_l = 0$ and the leading term has to be 1. \square

Theorem 5.2. Let $N \in \mathcal{H}_{n-1} \text{-mod}^{\text{fd}}$ be irreducible, $a = \pm 1$ and set

$$M := f_a N.$$

Then $\dim \text{End}_{\mathcal{H}_n}(M) \leq 2$.

Proof. By Frobenius reciprocity, $\text{End}_{\mathcal{H}_n}(M) \cong \text{Hom}_{\mathcal{H}_{n-1,1}}(N \boxtimes (a), \Delta_a M)$. Let $r-1 = \epsilon_a(N)$. By Lemma 5.1, the $(a_0, \underline{b}, a^{(r)})$ -eigenspace in M has at most twice the dimension of that in $N \boxtimes (a)$, therefore the socle of $\Delta_a M$ can contain at most 2 copies of $N \boxtimes (a)$. \square

Recall the automorphism σ fixing all generators except X_0 which it maps to its negative.

Corollary 5.3. If N in Lemma 5.2 satisfies $N \cong N^\tau$, $M = f_a N$ either has an irreducible cosocle or splits into a direct sum of two non-isomorphic irreducibles.

Proof. There are two cases to consider. The first case is the case $a = -1$. In this case, by $N \cong N^\tau$ and Corollary 2.7, we have

$$\begin{aligned} (\text{ind}_{n-1,1}^n N \boxtimes (-1))^\tau &\cong \text{ind}_{n-1,1}^n {}^{s_{n-1,0,n-1}}(N^\tau \boxtimes (-1)^\tau) \\ &\cong \text{ind}_{n-1,1}^n {}^{s_{n-1,0,n-1}}(N \boxtimes (-1)) \\ &\cong \text{ind}_{n-1,1}^n N^\sigma \boxtimes (-1). \end{aligned}$$

(where N^σ might or might not be isomorphic to N). Now τ fixes the X_i , duality doesn't affect eigenvectors and $\text{soc } f_{-1} N \cong (\tilde{f}_{-1} N^\sigma)^\tau$. Therefore the usual argument of $\text{Hom}_{\mathcal{H}_{n-1,1}}(N^\sigma \boxtimes (-1), \Delta_{-1} Q)$ being nonzero for every constituent of $\tilde{f}_{-1} N^\sigma$

together with the restriction on the dimension of $(\pm a_0, \underline{b}, -1^{(\epsilon_{-1}(N))})$ -eigenspaces from Lemma 5.1 yields that the socle of $f_{-1}N$ can have at most two constituents. In case it is irreducible, $\text{cosoc } f_{-1}N^\sigma$ and therefore also $\text{cosoc } f_{-1}N$ is irreducible. In case $\text{soc } f_{-1}N$ has two constituents, its $(\pm a_0, \underline{b}, -1^{(\epsilon_{-1}(N))})$ -eigenspace has to contain $1 \otimes N \boxtimes (-1)$ and therefore socle and cosocle have to coincide. Thus $f_{-1}N \cong M_1 \oplus M_2$ and by the restriction on the dimension of the endomorphism ring, these two irreducibles have to be nonisomorphic.

If $a = 1$, then $N \cong N^\tau$ implies $M \cong M^\tau$ since

$$\begin{aligned} M^\tau &\cong (\text{ind}_{n-1,1}^n N \boxtimes (a))^\tau \\ &\cong \text{ind}_{n-1,1}^n {}^{s_{n-1,0,n-1}}(N^\tau \boxtimes (a)^\tau) \\ &\cong \text{ind}_{n-1,1}^n {}^{s_{n-1,0,n-1}}(N \boxtimes (a)) \\ &\cong \text{ind}_{n-1,1}^n N \boxtimes (a), \end{aligned}$$

so in fact M is self-dual. Now the same argument about every constituent of the socle contributing $(a_0, \underline{b}, -1^{(\epsilon_{-1}(N))})$ -eigenspace as in the previous case shows that either the socle (and hence by self-duality the cosocle) must be irreducible or the socle must equal the cosocle. In the latter case the module itself is the direct sum of two non-isomorphic irreducibles. \square

Now we want to consider a class of modules where the results are as in the case $a \notin \{\pm 1\}$.

Lemma 5.4. *For all $N \in \mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$ and all $a \in F$,*

$$(\text{ind}_{n-1,1}^n N \boxtimes (a))^\sigma \cong \text{ind}_{n-1,1}^n N^\sigma \boxtimes (a).$$

Proof. Since $1 \otimes N \boxtimes (a)$ generates $\text{ind}_{n-1,1}^n N \boxtimes (a)$ as well as $(\text{ind}_{n-1,1}^n N \boxtimes (a))^\sigma$ as an \mathcal{H}_n -module it suffices to check that the twisted action on $1 \otimes N \boxtimes (a)$ is the same as the action on $1 \otimes N^\sigma \boxtimes (a)$. So let $h \in \mathcal{H}_n$ and write $h = \sum_{w \in D_{n-1,1}} T_w h_w$ for $h_w \in \mathcal{H}_{n-1,1}$. Then, as σ fixes all of $\mathcal{H}_n^{\text{fin}}$,

$$\begin{aligned} \sigma(h) \otimes u &= \sigma\left(\sum_{w \in D_{n-1,1}} T_w h_w\right) \otimes u \\ &= \sum_{w \in D_{n-1,1}} T_w \sigma(h_w) \otimes u \\ &= \sum_{w \in D_{n-1,1}} T_w \otimes \sigma(h_w) u \end{aligned}$$

for $u \in N \boxtimes (a)$, which proves the claim. \square

Theorem 5.5. *Let $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ be irreducible with $\text{ch } M \neq \text{ch } M^\sigma$ and let $\epsilon_{-1}(M) = r \geq 1$. Assume that there exists an irreducible submodule $N \boxtimes (-1)$ of $\Delta_{-1}M$ such that $N^\sigma \not\cong N = L(a_0, \dots, a_{n-r}, -1^{(r-1)})$ for some $(a_0, a_1, \dots, a_{n-r}) \in (F^\times \setminus \{\pm 1\})^{n-r+1}$. Then*

- (i) $\Delta_{-1(r)}M \cong K \boxtimes L^A(-1^{(r)})$ for some irreducible $K \in \mathcal{H}_{n-r}\text{-mod}^{\text{fd}}$ with $\text{ch } K \neq \text{ch } K^\sigma$;
- (ii) M is the irreducible cosocle of $\text{ind}_{n-r,r}^n K \boxtimes L^A(-1^{(r)})$;
- (iii) $\text{soc } \Delta_{-1}M \cong N \boxtimes (-1)$, i.e. $\tilde{e}_{-1}M \cong N$;
- (iv) M is the irreducible cosocle of the non-irreducible module $f_{-1}N$, in particular $\tilde{f}_{-1}N \cong M$;
- (v) M is uniquely determined by $\text{ch } M$ and in particular $M \cong M^\tau$.

Proof. Proceed by induction on r . The case $r = 1$ is easy: Take an irreducible submodule $N \boxtimes (-1)$ of $\Delta_{-1}M$ with $N^\sigma \not\cong N \cong L(a_0, \dots, a_{n-1})$. Then this will

also play the role of K in the statement and by the Mackey Theorem

$$[\Delta_{-1} f_{-1} N] = [N \boxtimes (-1)] + [N^\sigma \boxtimes (-1)],$$

whence $\tilde{f}_{-1} N$ is irreducible. On the other hand $f_{-1} N$ is not irreducible since $\text{ch } M \neq \text{ch } M^\sigma$ and $\Delta_{-1} M \cong N \boxtimes (-1)$ since some composition factor (and thus the composition factor $N^\sigma \boxtimes (-1)$) has to be taken up by $\text{soc } f_{-1} N$ by the same argument as in Lemma 5.3. N is uniquely determined by $\text{ch } N$ as $N \in \text{Rep}_{\neq \pm 1} \mathcal{H}_{n-1}$, and if $\text{ch } M = \text{ch } M'$, $\Delta_{-1} M' \cong N \boxtimes (-1)$, whence $M' \cong \tilde{f}_{-1} N \cong M$.

Now inductively assume that (i)-(v) hold for N in the hypothesis of the theorem, in particular that there exists some irreducible $K \not\cong K^\sigma \in \mathcal{H}_{n-r} \text{-mod}^{\text{fd}}$ determined by $\text{ch } K$ such that $\Delta_{-1(r-1)} N \cong K \boxtimes L^A(-1^{(r-1)})$, $N \cong \text{ind}_{n-r, r-1}^{n-1} K \boxtimes L^A(-1^{(r-1)})$ and $\text{ch } N$ determines N .

For the inductive step, we apply the Shuffle Lemma to compute a filtration of $\Delta_{-1(r)} \text{ind}_{n-1, 1}^n N \boxtimes (-1)$. $D_{(n-r, r), (n-1, 1)} = \{1, s_{n-r, n-1}, s_{n-r, 0, n-1}\}$, but considering formal characters we see that the subquotient of $\text{res}_{n-r, r}^n \text{ind}_{n-1, 1}^n N \boxtimes (-1)$ corresponding to $s_{n-r, n-1}$ does not have formal character values with an (-1) -tail of length r , as it is isomorphic to $\text{ind}_{n-r-1, 1, r}^{n-r, r} s_{n-r, n-1} (\text{res}_{n-r-1, r, 1}^{n-1, 1} N \boxtimes (-1))$. Thus it does not contribute to $\Delta_{-1(r)} \text{ind}_{n-1, 1}^n N \boxtimes (-1)$ and the latter only has subquotients isomorphic to

$$(5.1) \quad \text{ind}_{n-r, r-1, 1}^{n-r, r} (\Delta_{-1(r-1)} N) \boxtimes (-1)$$

and

$$(5.2) \quad \text{ind}_{n-r, 1, r-1}^{n-r, r} s_{n-r, 0, n-1} (\Delta_{-1(r-1)} N) \boxtimes (-1).$$

The use of $\Delta_{-1(r-1)}$ instead of $\text{res}_{n-r, r-1, 1}^{n-1, 1}$ in the Mackey formula is validated by the fact that no other summand of $\text{res}_{n-r, r-1}^{n-1} N$ can contribute composition factors to $\Delta_{-1(r)} \text{ind}_{n-1, 1}^n N \boxtimes (-1)$ for lack of (-1) s at the end. Now, by the assumptions on N , (5.1) is isomorphic to $K \boxtimes L^A(-1^{(r)})$ while (5.2) has the same formal character as $K^\sigma \boxtimes L^A(-1^{(r)})$ and is therefore isomorphic to $K^\sigma \boxtimes L^A(-1^{(r)})$. As in Lemma 5.3, the assumption on N and Corollary 2.7 yield

$$(\text{ind}_{n-1, 1}^n N \boxtimes (-1))^\tau \cong \text{ind}_{n-1, 1}^n N^\sigma \boxtimes (-1).$$

Now, if $\text{ind}_{n-1, 1}^n N \boxtimes (-1)$ were irreducible and therefore isomorphic to M , M^τ would also have to be isomorphic to $\text{ind}_{n-1, 1}^n N^\sigma \boxtimes (-1)$ and therefore $M^\tau \cong M^\sigma$, a contradiction since

$$\text{ch } M^\tau = \text{ch } M \neq \text{ch } M^\sigma.$$

Therefore, $\text{ind}_{n-1, 1}^n N \boxtimes (-1)$ is not irreducible. But since, for any quotient Q of $\text{ind}_{n-1, 1}^n N \boxtimes (-1)$ and in particular for any constituent of $\text{cosoc } \text{ind}_{n-1, 1}^n N \boxtimes (-1)$,

$$\begin{aligned} \text{Hom}_{\mathcal{H}_n}(\text{ind}_{n-r, r}^n K \boxtimes L^A(-1^{(r)}), Q) \\ \cong \text{Hom}_{\mathcal{H}_{n-r, r}}(K \boxtimes L^A(-1^{(r)}), \Delta_{-1(r)} Q) \end{aligned}$$

is nonzero and at the same time at most one-dimensional (as $K \boxtimes L^A(-1^{(r)})$ appears only once in $\Delta_{-1(r)} \text{ind}_{n-1, 1}^n N \boxtimes (-1)$), this shows that $\text{cosoc } \text{ind}_{n-1, 1}^n N \boxtimes (-1)$ is irreducible and therefore isomorphic to M , which proves (iv).

The same argument applied to $\text{ind}_{n-1, 1}^n N^\sigma \boxtimes (-1)$ shows that

$$\widetilde{M} = \text{soc } \text{ind}_{n-1, 1}^n N \boxtimes (-1) \cong \text{cosoc } \text{ind}_{n-1, 1}^n N^\sigma \boxtimes (-1)$$

is also irreducible and has $\epsilon_{-1}(\widetilde{M}) = r$. Therefore \widetilde{M} contributes the composition factor $K^\sigma \boxtimes L^A(-1^{(r)})$ to $\Delta_{-1(r)} \text{ind}_{n-1, 1}^n N \boxtimes (-1)$ and

$$\Delta_{-1(r)} M \cong K \boxtimes L^A(-1^{(r)}),$$

so (i) holds.

To see that $N \boxtimes (-1)$ is actually the whole of $\text{soc } \Delta_{-1}M$ it suffices to consider the fact that, if $N' \boxtimes (-1)$ is an irreducible submodule of $\Delta_{-1}M$, it has to satisfy $\epsilon_{-1}(N') = r - 1$ as in this case

$$\text{Hom}_{\mathcal{H}_n}(\text{ind}_{n-1,1}^n N' \boxtimes (-1), M) \neq 0.$$

But then $0 \neq \Delta_{-1(r-1)} N' \cong K' \boxtimes L^A(-1^{(r-1)})$ whence, by transitivity of induction, $\text{ind}_{n-r,r}^n K' \boxtimes L^A(-1^{(r)})$ projects onto M or equivalently

$$K' \boxtimes L^A(-1^{(r)}) \hookrightarrow \Delta_{-1(r)} M,$$

a contradiction, whence (iii) follows.

Now, suppose some M' has the same formal character as M . Then $\Delta_{-1(r)} M'$ has the same formal character as $\Delta_{-1(r)} M$. By virtue of K being uniquely determined by its formal character, we obtain

$$\Delta_{-1(r)} M' \cong K \boxtimes L^A(-1^{(r)}),$$

so (v) will follow if we can show (ii), namely that $\text{cosoc } \text{ind}_{n-r,r}^n K \boxtimes L^A(-1^{(r)})$ is irreducible. It certainly does not contain several copies of M since

$$\begin{aligned} \text{Hom}_{\mathcal{H}_{n-r,r}}(K \boxtimes L^A(-1^{(r)}), \Delta_{-1(r)} M) \\ \cong \text{Hom}_{\mathcal{H}_n}(\text{ind}_{n-r,r}^n K \boxtimes L^A(-1^{(r)}), M) \end{aligned}$$

is one-dimensional, so assume it contains an irreducible submodule M' not isomorphic to M . But again, choosing $N' \boxtimes (-1)$ in the socle of $\Delta_{-1}M'$ shows that necessarily

$$\Delta_{-1(r-1)} N' \cong K \boxtimes L^A(-1^{(r-1)}),$$

whence (by the irreducibility of $\text{cosoc } \text{ind}_{n-r,r-1}^{n-1} K \boxtimes L^A(-1^{(r-1)})$) $N' \cong N$ and we're done. \square

Remark 5.6. By an additional induction over the number of -1 -strings in the tuple labelling N in Theorem 5.5, which mainly uses that applying \tilde{f}_a for $a \neq \pm 1$ in between the -1 -strings won't affect the uniqueness of formal characters, we can omit the hypothesis that the a_i ($1 \leq i \leq n - r$) are all distinct from -1 .

Corollary 5.7. *In the situation of Theorem 5.5,*

$$\text{soc } \text{ind}_{n-1,1}^n N \boxtimes (-1) \cong \text{cosoc } \text{ind}_{n-1,1}^n N^\sigma \boxtimes (-1) \cong M^\sigma.$$

We have seen that for $a = -1$, as long as $M \not\cong M^\sigma$, everything works as in the case $a \neq \pm 1$, and $M \not\cong M^\sigma$ is satisfied until, for the first time, the induced module $\text{ind}_{n-1,1}^n N \boxtimes (-1)$ is irreducible and therefore isomorphic to its σ -conjugate. The next lemma explicitly computes eigenspaces in induced modules.

Lemma 5.8. *Let $N \cong N^\tau \in \mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$ be irreducible, $a \in \{\pm 1\}$ and $\epsilon_a(N) = r - 1$. Set $M := f_a N$. The generalized $(a^{(r)})$ -eigenspace for X_{n-r+1}, \dots, X_n in M is contained in the socle and cosocle of M , thus all other composition factors K have $\epsilon_a(K) \leq r - 1$.*

Proof. By the Mackey Theorem 2.3 ,

$$\begin{aligned} [\Delta_{a^{(r)}} M] &= [\text{ind}_{n-r,r-1,1}^{n-r,r} \Delta_{a^{(r-1)}} N \boxtimes (a)] \\ &\quad + [\text{ind}_{n-r,1,r-1}^{n-r,r} s_{n-r,0,n-1} (\Delta_{a^{(r-1)}} N \boxtimes (a))] \end{aligned}$$

and as in Lemma 5.1 we know that the $(a^{(r)})$ -eigenspace of (X_{n-r+1}, \dots, X_n) in the submodule $\text{ind}_{n-r,r-1,1}^{n-r,r} \Delta_{a^{(r-1)}} N \boxtimes (a)$, which is by Theorem 2.3 contained in the socle of $\Delta_{a^{(r)}} M$, is contained in $1 \otimes \Delta_{a^{(r-1)}} N \boxtimes (a)$. Under the restriction of the projection of M onto its cosocle, this certainly doesn't map to zero, whence we have

an injection of $\text{ind}_{n-r, r-1, 1}^{n-r, r} \Delta_{a(r-1)} N \boxtimes (a)$ into $\Delta_{a(r)} \text{cosoc } M$. Since M is self-dual in case $a = 1$ or $a = -1$ and $N \cong N^\sigma$ and $M^\tau \cong M^\sigma$ and therefore $\text{soc } M \cong (\text{cosoc } M^\sigma)^\tau$ in the remaining case, the generalized $(a^{(r)})$ -eigenspace $\Delta_{a(r)} \text{soc } M$ of the socle must have the same dimension, thus exhausting all of $\Delta_{a(r)} M$. \square

We will now give an example where indeed the functor \tilde{f}_1 does not produce an irreducible module.

Example 5.9. We apply the functor f_1 to the two-dimensional irreducible module $L(a_0, q^2) \cong \text{ind}_{\mathcal{P}_1}^{\mathcal{H}_1}(a_0, q^2) \in \mathcal{H}_1\text{-mod}^{\text{fd}}$ with basis $\{w_1, w_2\}$ on which T_0, X_0, X_1 act by matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & (p - p^{-1}) \end{pmatrix}, \quad \begin{pmatrix} a_0 & -(p - p^{-1})a_0q^2 \\ 0 & a_0q^2 \end{pmatrix}, \quad \begin{pmatrix} q^2 & (p - p^{-1})(q^2 + 1) \\ 0 & q^{-2} \end{pmatrix}$$

respectively. Since $w_1 \in L(a_0, q^2)$ is an (a_0, q^2) -eigenvector, Lemma 1.5 implies that, in the induced module $M := f_1 L(a_0, q^2) = \mathcal{H}_2 \otimes_{\mathcal{H}_{1,1}} L(a_0, q^2) \boxtimes (1)$, we find an $(a_0, 1, q^2)$ -eigenvector $v_1 := (T_1 + q^{-1})w_1$. Again by Lemma 1.5, the vector $(T_0 - q^2 T_0^{-1})w_1 \in L(a_0, q^2)$ is an $(a_0 q^2, q^{-2})$ -eigenvector for (X_0, X_1) , and $v_2 := (T_1 - q)(T_0 - q^2 T_0^{-1})w_1$ is an $(a_0 q^2, 1, q^{-2})$ -eigenvector in M . It is then easy to calculate that v_1 and v_2 both generate 4-dimensional submodules with trivial intersection which are both irreducible and isomorphic to $L(a_0, 1, q^2)$ and $L(a_0 q^2, 1, q^{-2})$ respectively. Therefore

$$\tilde{f}_1 L(a_0, q^2) \cong f_1 L(a_0, q^2) \cong L(a_0, 1, q^2) \oplus L(a_0 q^2, 1, q^{-2}).$$

6. FACTS ABOUT REPRESENTATIONS OF THE AFFINE HECKE ALGEBRA OF TYPE A

At this point it is convenient to recall some important facts from the representation theory of affine Hecke algebras of type A. The results in this section are compiled from [2], [1], [14], [15], [8] and [7].

For nonzero $\lambda \in F$, define the full subcategory $\text{Rep}_\lambda \mathcal{H}_n^A$ of $\mathcal{H}_n^A\text{-mod}^{\text{fd}}$ to consist of those modules where all eigenvalues of the generators \mathcal{R}_n are from $I_\lambda^+ := \{\lambda q^{2i} | i \in \mathbb{Z}\}$. The significance of the "plus" will become clear later on when we move to type B.

The $\text{Rep}_\lambda \mathcal{H}_n^A$ are equivalent categories for all $\lambda \in F$, and for $I_\lambda \neq I_{\lambda'}$, there are no nontrivial extensions between modules in $\text{Rep}_\lambda \mathcal{H}_n^A$ and $\text{Rep}_{\lambda'} \mathcal{H}_n^A$ for $I_\lambda \neq I_{\lambda'}$ and $\text{ind}_{\mathcal{H}_{n_1}^A \otimes \mathcal{H}_{n_2}^A}^{\mathcal{H}_{n_1+n_2}^A} M \boxtimes N$, for irreducible M and N in $\text{Rep}_\lambda \mathcal{H}_{n_1}^A$ and $\text{Rep}_{\lambda'} \mathcal{H}_{n_2}^A$ respectively, is always irreducible. The irreducible modules in $\text{Rep}_\lambda \mathcal{H}_n^A$ are well-understood and have a nice combinatorial description.

Call the sequence $\Gamma_{(i, i+k)} = (aq^{2i}, aq^{2(i+1)}, \dots, aq^{2(i+k)})$ a segment and denote by $L_{\Gamma_{(i, i+k)}} = L^A(aq^{2i}, aq^{2(i+1)}, \dots, aq^{2(i+k)})$ the one-dimensional representation of \mathcal{H}_{k+1}^A on which all T_j , for $1 \leq j \leq k$ act as q and X_l acts as $aq^{2(i+l-1)}$ for $1 \leq j \leq k+1$. Define a multisegment $\Gamma = (\Gamma_1, \dots, \Gamma_m)$ to be a concatenation of several segments and denote by $L_\Gamma := L_{\Gamma_1} \boxtimes \dots \boxtimes L_{\Gamma_m}$ the one-dimensional representation for the tensor product of the corresponding algebras. The length of a multisegment is the sum of the lengths of the contained segments. We have two different orderings on multisegments, the so-called right and left orders.

In the right order, $\Gamma_{(i, i+k)} > \Gamma_{(j, j+l)}$ if $i > j$ or if $i = j$ and $l > k$.

In the left order, $\Gamma_{(i, i+k)} > \Gamma_{(j, j+l)}$ if $i + k > j + l$ or $i + k = j + l$ and $j > i$.

Bernstein and Zelevinski [2] showed that there is a one-to-one correspondence between ordered multisegments of length n and irreducible modules for the affine Hecke algebra \mathcal{H}_n^A of type A. This correspondence is given by inducing L_Γ for a multisegment Γ up to \mathcal{H}_n^A and taking the – always irreducible – cosocle of this

induced module which is independent of whether we have chosen Γ in right or left order. Denote this cosocle by N_Γ .

There is also a combinatorial description of some form of branching rules. Since the index of \mathcal{H}_{n-1}^A in \mathcal{H}_n^A is infinite, we substitute normal induction by functors

$$\text{ind}_a^A := \text{ind}_{\mathcal{H}_{n-1,1}^A}^{\mathcal{H}_n^A} - \boxtimes(a) : \text{Rep}_\lambda \mathcal{H}_{n-1}^A \longrightarrow \text{Rep}_\lambda \mathcal{H}_n^A$$

for every $a \in I_\lambda^+$. For irreducible N in $\text{Rep}_\lambda \mathcal{H}_{n-1}^A$, the cosocle of $\text{ind}_a^A N$ is irreducible and we denote this by $\tilde{f}_a^A N$. Dually, we define

$$\tilde{e}_a^A N := \text{res}_{\mathcal{H}_{n-1}^A}^{\mathcal{H}_{n-1,1}^A} \text{soc } \Delta_a N$$

where $\Delta_a N$ is the generalized a -eigenspace of X_n in N as in type B . For irreducible N , $\tilde{e}_a^A N$ is again an irreducible module. As in type B , we define $\epsilon_a(N)$ to be the largest number r such that $\Delta_{a^{(r)}} N \neq 0$. Since the definitions of Δ_a and ϵ_a involve the action of the lattice, we keep the notation from type B and do not mark the symbols with an A . Analogously, define

$$\text{ind}_a^{*A} := \text{ind}_{\mathcal{H}_{1,n-1}^A}^{\mathcal{H}_n^A} (a) \boxtimes - : \text{Rep}_\lambda \mathcal{H}_{n-1}^A \rightarrow \text{Rep}_\lambda \mathcal{H}_n^A,$$

$$\tilde{f}_a^{*A} N := \text{cosoc } \text{ind}_a^{*A} N \quad \text{and} \quad \tilde{e}_a^{*A} N := \text{res}_{\mathcal{H}_{n-1}^A}^{\mathcal{H}_{1,n-1}^A} \text{soc } \Delta_a^* N,$$

where $\Delta_a^* N$ is the generalized a -eigenspace of X_1 on N which is, just as $\Delta_a N$, an $\mathcal{H}_{1,n-1}^A$ -submodule of N since X_1 commutes with T_2, \dots, T_n . Both $\tilde{f}_a^{*A} N$ and $\tilde{e}_a^{*A} N$ are irreducible if N is. Lastly, $\epsilon_a^*(N)$ is defined as the maximal r such that $\Delta_{a^{(r)}}^* N \neq 0$.

For the irreducible module N_Γ in $\text{Rep}_\lambda \mathcal{H}_n^A$ there are combinatorial algorithms to compute $\tilde{f}_a^A N_\Gamma$, $\tilde{f}_a^{*A} N_\Gamma$, $\tilde{e}_a^A N_\Gamma$, $\tilde{e}_a^{*A} N_\Gamma$, $\epsilon_a(N_\Gamma)$ and $\epsilon_a^*(N_\Gamma)$ which will be described below.

To compute $\epsilon_a(N_\Gamma)$, write down the multisegment Γ in right order. Then, write a $+$ for every segment ending on aq^{-2} and a $-$ for every multisegment ending on a . In the resulting sequence of plus and minus signs successively cancel out all subsequences of the form $-+$ until the leftover sequence is of the form $+\dots + - \dots -$. The number of uncanceled $-$ signs is $\epsilon_a(N_\Gamma)$. If we replace the segment $(aq^{-2i}, \dots, aq^{-2}, a)$ which contributed the leftmost uncanceled $-$, by $(aq^{-2i}, \dots, aq^{-2})$ we get the multisegment corresponding to $\tilde{e}_a^A N_\Gamma$. Denote this multisegment by $\tilde{e}_a^A \Gamma$. In case there is no $-$ sign left after cancellation $\tilde{e}_a^A N_\Gamma = 0$. If we replace the segment $(aq^{-2k}, \dots, aq^{-2})$ which contributed the rightmost uncanceled $+$, by $(aq^{-2k}, \dots, aq^{-2}, a)$ we get the multisegment $\tilde{f}_a^A \Gamma$ corresponding to $\tilde{f}_a^A N_\Gamma$. If there is no $+$ left after cancellation, we add a new segment (a) to Γ to obtain $\tilde{f}_a^A N_\Gamma$.

To compute $\epsilon_a^*(N_\Gamma)$, write down the multisegment Γ in left order. Then, write a $+$ for every segment starting on aq^2 and a $-$ for every multisegment starting on a . In the resulting sequence of plus and minus signs successively cancel out all subsequences of the form $+ -$ until the leftover sequence is of the form $- \dots - + \dots +$. The number of uncanceled $-$ signs is $\epsilon_a^*(N_\Gamma)$. If we replace the segment $(a, aq^2, \dots, aq^{2i})$ which contributed the leftmost uncanceled $-$, by (aq^2, \dots, aq^{2i}) we get the multisegment corresponding to $\tilde{e}_a^{*A} N_\Gamma$. Denote this multisegment by $\tilde{e}_a^{*A} \Gamma$. If there is no $-$ sign left after cancellation $\tilde{e}_a^{*A} N_\Gamma = 0$. If we replace the segment (aq^2, \dots, aq^{2k}) which contributed the rightmost uncanceled $+$, by $(a, aq^2, \dots, aq^{2k})$ we get the multisegment $\tilde{f}_a^{*A} \Gamma$ corresponding to $\tilde{f}_a^{*A} N_\Gamma$. If there is no $+$ left after cancellation, we add a new segment (a) to Γ to obtain $\tilde{f}_a^{*A} N_\Gamma$. Moreover,

$$\tilde{f}_a^{*A} N_\Gamma \cong \text{soc } \text{ind}_a^A N_\Gamma \quad \text{and} \quad \tilde{f}_a^A N_\Gamma \cong \text{soc } \text{ind}_a^{*A} N_\Gamma.$$

7. SUBCATEGORIES WITH TYPE-A BEHAVIOR

In this Chapter we investigate certain subcategories of $\mathcal{H}_n\text{-mod}^{\text{fd}}$ which behave very similarly to the situation in type A . We define $\text{Rep}_\lambda \mathcal{H}_n$ for fixed nonzero $\lambda \in F$ to be the full subcategory of $\mathcal{H}_n^R\text{-mod}^{\text{fd}}$ where all eigenvalues of \mathcal{R}_n are from the set

$$I_\lambda := \{\lambda q^{2i}, \lambda^{-1} q^{2i} | i \in \mathbb{Z}\}.$$

In this section, we consider the cases where $p^2, \pm q, \pm 1 \notin I_\lambda$. In these cases $I_\lambda^+ := \{\lambda q^{2i} | i \in \mathbb{Z}\}$ and $I_\lambda^- := \{\lambda^{-1} q^{2i} | i \in \mathbb{Z}\}$ are disjoint. Since we exclude the eigenvalue -1 , Lemmas 1.1 and 1.3 guarantee that we can work with the algebra \mathcal{H}_n^R instead of \mathcal{H}_n since all irreducibles for \mathcal{H}_n in the analogous subcategories are obtained by extending the action to \mathcal{H}_n with an arbitrary new eigenvalue for X_0 . Working with \mathcal{H}_n^R here is more convenient since we want to exploit the subalgebra \mathcal{H}_n^A , which has finite index in \mathcal{H}_n^R but not in \mathcal{H}_n .

First, we would like to give a very general result on the formal characters of an \mathcal{H}_n^R -module obtained by inducing from \mathcal{H}_n^A .

Lemma 7.1. *Let $N \in \mathcal{H}_n^A\text{-mod}^{\text{fd}}$ be irreducible. Set $M := \text{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} N$. Then*

$$\epsilon_a(M) \leq \epsilon_a(N) + \epsilon_{a^{-1}}^*(N).$$

Proof. By the Shuffle Lemma, we obtain the formal character of M by taking all formal characters of N and then successively inverting the first entries of the tuple and moving them to the rear, see Section 3 for an example. From this, one easily sees that the maximal number of a 's at the end of a tuple from $\text{ch } M$ is less or equal to the maximal number of a 's at the end of a tuple from $\text{ch } N$ plus the maximal number of a^{-1} 's at the beginning of a tuple from $\text{ch } N$ that get inverted and moved to the rear. \square

Note that on \mathcal{H}_n^A there is an algebra antiautomorphism κ given by $T_i \mapsto T_{n-i}$ and $X_i \mapsto X_{n+1-i}^{-1}$ inducing a duality on $\mathcal{H}_n^A\text{-mod}^{\text{fd}}$. It is easy to check on the generators that κ is the composite of first taking the τ -dual and then twisting with the longest coset representative $d = s_{0,1,0} \cdots s_{j,0} \cdots s_{n-1,0}$. Now we compute the κ -dual of an irreducible $N_\Gamma \in \text{Rep}_\lambda \mathcal{H}_n^A$, where Γ consists of segments $\Gamma_1, \dots, \Gamma_r$ of length n_1, \dots, n_r respectively:

$$\begin{aligned} N_\Gamma^\kappa &\cong {}^d(N_\Gamma^\tau) \\ &\cong {}^d(N_\Gamma) \\ &= \text{cosoc} {}^d(\mathcal{H}_n^A \otimes_{\mathcal{H}_{n_1}^A} \cdots \otimes_{\mathcal{H}_{n_r}^A} L_\Gamma) \\ &\cong \text{cosoc } \mathcal{H}_n^A \otimes {}^d(\mathcal{H}_{n_1}^A \otimes \cdots \otimes \mathcal{H}_{n_r}^A) {}^d L_\Gamma \\ &\cong \text{cosoc } \mathcal{H}_n^A \otimes_{\mathcal{H}_{n_r}^A} \cdots \otimes_{\mathcal{H}_{n_1}^A} L_{\bar{\Gamma}} \\ &\cong N_{\bar{\Gamma}}, \end{aligned}$$

where $\bar{\Gamma}$ is the multisegment $\bar{\Gamma}_r, \dots, \bar{\Gamma}_1$, and the segment $\bar{\Gamma}_j$, for a segment $\Gamma_j = (a, \dots, aq^{2k})$, is defined as $\bar{\Gamma}_j = (a^{-1}q^{-2k}, \dots, a^{-1})$. The second isomorphism uses the fact that in type A all irreducibles are self-dual under τ -duality.

Lemma 7.2. *Let $N_\Gamma \in \text{Rep}_{\lambda^{-1}} \mathcal{H}_n^A$ be irreducible. Then $M_\Gamma := \text{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} N_\Gamma \in \text{Rep}_\lambda \mathcal{H}_n$ is irreducible.*

Proof. Since all formal characters of N_Γ have entries in I_λ^- and $I_\lambda^- \cap I_\lambda^+ = \emptyset$, the Shuffle Lemma implies that the only summands in the formal character of M_Γ exclusively containing entries from I_λ^- are those afforded by the coset representative

1. Thus

$$\mathrm{Hom}_{\mathcal{H}_n^R}(M_\Gamma, \mathrm{cosoc} M_\Gamma) \cong \mathrm{Hom}_{\mathcal{H}_n^A}(N_\Gamma, \mathrm{res}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} \mathrm{cosoc} M_\Gamma) \cong F,$$

whence the cosocle of M_Γ is irreducible. Since by Corollary 2.7

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}_n^R}(\mathrm{soc} M_\Gamma, M_\Gamma) &\cong \mathrm{Hom}_{\mathcal{H}_n^R}(M_\Gamma^\tau, \mathrm{cosoc} M_\Gamma^\tau) \\ &\cong \mathrm{Hom}_{\mathcal{H}_n^R}(\mathrm{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} d(N_\Gamma^\tau), \mathrm{cosoc} \mathrm{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} d(N_\Gamma^\tau)) \\ &\cong \mathrm{Hom}_{\mathcal{H}_n^R}(\mathrm{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} N_{\overline{\Gamma}}, \mathrm{cosoc} \mathrm{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} N_{\overline{\Gamma}}) \\ &\cong \mathrm{Hom}_{\mathcal{H}_n^A}(N_{\overline{\Gamma}}, \mathrm{res}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} \mathrm{cosoc} \mathrm{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} N_{\overline{\Gamma}}) \\ &\cong F \end{aligned}$$

by the same argument as above, the socle of M_Γ is also simple and contains the generalized simultaneous eigenspace of \mathcal{R}_n where all eigenvalues of the X_i are from I_λ^+ .

Now take a (b_1, \dots, b_n) -eigenvector $u \in M_\Gamma$ such that b_1, \dots, b_n are all in I_λ^+ . Then

$$\begin{aligned} v := & (T_0 - b_n T_0^{-1})(T_1 - b_{n-1} b_n T_1^{-1})(T_0 - b_{n-1} T_0^{-1}) \\ & \cdots (T_{n-j} - b_j b_n T_{n-j}^{-1}) \cdots (T_0 - b_j T_0^{-1}) \\ & \cdots (T_{n-1} - b_1 b_n T_{n-1}^{-1}) \cdots (T_1 - b_1 b_2 T_1^{-1})(T_0 - b_1 T_0^{-1})u \end{aligned}$$

is a $(b_n^{-1}, \dots, b_1^{-1})$ -eigenvector by Lemma 1.5 since $b_j \notin \{p^{\pm 2}, 1\}$ and $b_j b_k \notin \{q^{\pm 2}\}$. Now all b_i^{-1} belong to I_λ^- , therefore $v \in 1 \otimes N_\Gamma$ and generates M_Γ . Therefore any element in the socle of M_Γ generates the whole of M_Γ , so it must be irreducible. \square

Lemma 7.3. *Let $N_\Gamma \in \mathrm{Rep}_{\lambda^{-1}} \mathcal{H}_n^A$ be irreducible.*

$$\text{Then } \epsilon_a(\mathrm{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} N_\Gamma) = \begin{cases} \epsilon_a(N_\Gamma) & \text{if } a \in I_\lambda^- \\ \epsilon_{a^{-1}}^*(N_\Gamma) & \text{if } a \in I_\lambda^+ \end{cases}$$

Proof. This follows directly from the proof of Lemma 7.1 and the fact that $I_\lambda^- \cap I_\lambda^+ = \emptyset$. \square

Lemma 7.4. *Let $M_\Gamma \in \mathrm{Rep}_\lambda \mathcal{H}_n$ be defined as in Lemma 7.2 and $a \in I_\lambda$. Then*

$$\begin{aligned} \text{(i)} \quad \tilde{f}_a M_\Gamma &= \begin{cases} M_{\tilde{f}_a^A \Gamma} & \text{if } a \in I_\lambda^- \\ M_{\tilde{f}_{a^{-1}}^{*A} \Gamma} & \text{if } a \in I_\lambda^+ \end{cases} \\ \text{(ii)} \quad \tilde{e}_a M_\Gamma &= \begin{cases} M_{\tilde{e}_a^A \Gamma} & \text{if } a \in I_\lambda^- \\ M_{\tilde{e}_{a^{-1}}^{*A} \Gamma} & \text{if } a \in I_\lambda^+ \end{cases} \end{aligned}$$

Proof. (i) Without loss of generality we assume $a \in I_\lambda^-$ and compute $\tilde{f}_a M_\Gamma$ and $\tilde{f}_{a^{-1}} M_\Gamma = \mathrm{soc} \mathrm{ind}_{H_{n,1}^R}^{\mathcal{H}_{n+1}^R} M_\Gamma \boxtimes (a)$.

We know that

$$\begin{aligned} \mathrm{ind}_{H_{n,1}^R}^{\mathcal{H}_{n+1}^R} M_\Gamma \boxtimes (a) &\cong \mathrm{ind}_{H_{n,1}^R}^{\mathcal{H}_{n+1}^R} (\mathrm{ind}_{H_n^A}^{\mathcal{H}_n^R} N_\Gamma) \boxtimes (a) \\ &\cong \mathrm{ind}_{H_{n+1}^A}^{\mathcal{H}_{n+1}^R} \mathrm{ind}_{H_{n,1}^A}^{\mathcal{H}_{n+1}^A} N_\Gamma \boxtimes (a), \end{aligned}$$

so, as every composition factor in $\mathrm{ind}_{H_{n,1}^A}^{\mathcal{H}_{n+1}^A} N_\Gamma \boxtimes (a)$ is in $\mathrm{Rep}_{\lambda^{-1}} \mathcal{H}_{n+1}^A$ and therefore yields an irreducible subquotient of $\mathrm{ind}_{H_{n,1}^R}^{\mathcal{H}_{n+1}^R} M_\Gamma \boxtimes (a)$ upon induction to type B , $\mathrm{ind}_{H_{n,1}^R}^{\mathcal{H}_{n+1}^R} M_\Gamma \boxtimes (a)$ has the same number of composition factors as $\mathrm{ind}_{H_{n,1}^A}^{\mathcal{H}_{n+1}^A} N_\Gamma \boxtimes (a)$,

labeled by the same multisegments. Since socle and cosocle of $\text{ind}_{H_{n,1}^R}^{\mathcal{H}_{n+1}^R} M_\Gamma \boxtimes (a)$ are irreducible, they have to coincide with

$$\text{ind}_{H_{n+1}^A}^{\mathcal{H}_{n+1}^R} \text{soc} \text{ind}_{H_{n,1}^A}^{\mathcal{H}_{n+1}^A} N_\Gamma \boxtimes (a)$$

and

$$\text{ind}_{H_{n+1}^A}^{\mathcal{H}_{n+1}^R} \text{cosoc} \text{ind}_{H_{n,1}^A}^{\mathcal{H}_{n+1}^A} N_\Gamma \boxtimes (a)$$

respectively. The fact that $N_{\tilde{f}_a^* A \Gamma} = \text{soc} \text{ind}_{H_{n,1}^A}^{\mathcal{H}_{n+1}^A} N_\Gamma \boxtimes (a)$ implies that

$$\tilde{f}_{a^{-1}} M_\Gamma \cong \text{ind}_{H_{n+1}^A}^{\mathcal{H}_{n+1}^R} N_{\tilde{f}_a^* A \Gamma}$$

for $a \in I_\lambda^-$, which completes the proof of (i).

(ii) follows directly from Lemma 4.9. \square

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